A survey on cycles and chaos (part II)

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Abstract: This paper is an extension of a previous publication in the journal *Historical Social Research* (Vol. 26, No. 4, 2001, p. 208-219). Our treatment begins with a simple presentation of the basic notions of chaos, and then describes the related econometric tools.

1. Introduction

The term complex economic dynamics is used to designate deterministic economic models whose trajectories exhibit irregular (nonperiodic) fluctuations or endogenous phase switching. The first property includes chaotic trajectories that give bounded fluctuations which are sensitive to perturbations. The second means that the equations governing change in system states switch from time to time according to intrinsic rules. Or it means that distinct types of qualitative behavior, such as growth, oscillation or decay, are exhibited in different subsets of the state space; the system equations restricted to a given subset then appear to have a different nature than their restriction to other subsets, so that each such restriction yields an identifiable regime.

Chaotic processes have many very interesting properties, only a few of which need to be mentioned here. The first is the existence of attractors. Suppose that many terms of the process have been generated, so that t is large, and let $x_{t,m}$ be a vector of m adjacent values $(x_t, x_{t-1},\ldots, x_{t-m+1})$. For a certain value of m, called the embedding dimension, $x_{t,m}$ will always lie on a particular subset of the m-dimensional space, called the attractor of the process. A chaotic process is in a sense simple if its embedding dimension is low (say one to three) and

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is complicated if it is high. For example, the logistic map ($x_t = \mu x_{t-1}(1-x_{t-1})$) has a dimension of one whereas a white noise process have very high dimension.

The empirical testing in economics and finance finds plenty of evidence for nonlinearity but none for low dimensional chaos. This suggests that there are stochastic shocks occurring somewhere in the economy, so one has to ask how this fits in with the chaos theory. Experiments have also shown that adding a little white noise to a low dimensional chaotic signal, makes the deterministic chaos extremely difficult to detect in short series. Thus emerges the interest of introducing the approach of stochastic chaos. As it has been underlined by Chan and Tong (1994), it is more realistic to model economic or financial date with a nonlinear deterministic process perturbed by dynamical noise.

The purpose of the paper is to present the recent developed tests for chaos: the correlation dimension, the Lyapunov exponents and the surrogate date tests.

2. The correlation dimension test

The correlation dimension was introduced by Grassberger and Procaccia (1983). The correlation dimension is based on the idea that if an attractor is chaotic, then two points (X_i, X_j) starting at different positions will be dynamically uncorrelated as a result of the property of sensitive dependence on initial conditions. However, since the points are on an attractor, they can approach each other but can never intersect.

The correlation between points on an attractor can be defined in term of spatial correlation that is formally measured by the Euclidean distance.

Let $\{X_t\}$, t = 1,2…,T be a sample from a strictly stationary process. The time series $\{X_t\}$ can be "embedded" in a m-space by constructing "m-histories". The correlation dimension can be calculated from the correlation integral given by:

$$C(\varepsilon, m, T_m) = \frac{1}{T_m(T_m-1)} \sum_{i,j=1}^{T_m} H\left(\left\| X_i - X_j \right\| \right) \quad i \neq j \qquad (1)$$

as defined in the Part I (Diebolt and Kyrtsou, 2001).

The use of an Euclidean norm for computing the correlation dimension is considered not to be too restrictive. Brock (1986, theorem 2.4) has proved that the correlation dimension is independent of the choice of norm.

Let the correlation integral measure the fraction of total number of pairs (xi, xi+1,…, xi+m-1), (xj, xj+1,…, xj+m-1), such that the distance between them is no more than ε. The correlation dimension can be defined as follows:

$$d_c = \lim_{\varepsilon \to 0} \frac{\ln C(\varepsilon, m)}{\ln \varepsilon} \qquad (2)$$

For the small values of ε, Grassberger and Procaccia (1983) establish that the spatial correlation C(ε,m) grows according to the power law:

If $d_m = \lim_{\varepsilon \to 0} \frac{\ln C(\varepsilon, m)}{\ln \varepsilon}$, then $\ln C(\varepsilon,m) \approx d_m \ln\varepsilon \Leftrightarrow \ln C(\varepsilon,m) \approx \ln \varepsilon^{d_m} \Leftrightarrow C(\varepsilon,m) \approx \varepsilon^{d_m}$, and C(ε,m) grows exponentially.

It is necessary to notice that when the embedding dimension m increases, the dimension d_m is reached, such that d^*_c is the estimate of the true correlation:

$$d^*_c = \lim_{m \to \infty} d_m \tag{3}$$

The method of the correlation dimension represents a very important diagnostic procedure for distinguishing between determinism and stochasticity. If d_m tends to be a constant as m increases, then d_m yields an estimate of the correlation dimension of the attractor, namely d^*_c. In this case, the time series are consistent with deterministic behavior. If d_m increases without bound as m increases, this suggests that the underlying series are stochastic[1].

3. The Lyapunov exponent test

The Lyapunov exponent method can be employed to determine if a process is chaotic. The approach is based on the idea that the distance between two points is described by the largest Lyapunov exponent. The Lyapunov exponents measure the average rate of contraction (when negative) or expansion (when positive) of the trajectories on the entire attractor. They can be positive or negative, but at least one exponent must be positive for an attractor to be classified as chaotic. If the distance between the trajectories grows exponentially, this is evidence of chaos since it shows that the process exhibits sensitive dependence to initial conditions.

Thus, where λ is the largest Lyapunov exponent, the criterion is:

Noisy chaos or stochasticity if $\lambda < 0$,
chaos if $\lambda > 0$

In the n-dimensional case, where yt+1 = f(yt) (3) with $t \in T$, $y \in Rn$, the Lyapunov exponent λ is defined (Lorenz, 1989) by $\lambda^{(T)}=(1/T)\log_2(\Lambda^{(T)})$, where $\Lambda^{(T)}$ are the eigenvalues of the n-dimensional Jacobian matrix $J^{(T)}$. In

[1] Ruelle (1990) argues that a chaotic series can only be distinguished if it has a correlation dimension well below $2\log_{10}T$, where T is the size of the data set, suggesting that with economic time series the correlation dimension can only distinguish low dimensional chaos from high dimensional stochastic processes.

general, all Lyapunov exponents can be calculated according to the following equation (see Wolf et al., 1985):

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \log_2(\Lambda_i^{(T)}) \qquad (4)$$

When applying this method to financial-price series, many authors confirme the difficulty of pollution from high frequency noise. The largest Lyapunov exponent λ tends to be greater than the true exponent and its convergence to a value appears difficult or even impossible.

3.1 Kantz algorithm (1994)

Kantz (1994) has tried to solve this problem by constructing a new algorithm for the estimation of λ. Similar to Wolf et al. (1985), he makes use of the fact that the distance between two trajectories typically increases with a rate given by the maximal Lyapunov exponent. This divergence rate of trajectories naturally fluctuates along the trajectory, with the fluctuations given by the spectrum of effective Lyapunov exponents. The maximal exponent λ_τ is defined to be:

$$\lambda_\tau(t) = \lim_{\varepsilon\to 0} \frac{1}{\tau} \ln\left(\frac{|\chi(t+\tau) - \chi_\varepsilon(t+\tau)|}{\varepsilon}\right) \qquad (5)$$

where $\chi(t)$ is the time evolution of some initial condition $\chi(0)$ in an appropriate state space, t is time, and τ is relative time referring to the time index of the starting point, and $\varepsilon = |\chi(0)-\chi_\varepsilon(0)|$. $\chi(t)-\chi_\varepsilon(t) = \varepsilon\omega_u(t)$, where $\omega_u(t)$ is the local eigenvector associated with the maximal Lyapunov exponent λ_{max}. By definition the average of $\lambda_\tau(t)$ along the trajectory is the true Lyapunov exponent.

The method of Kantz requires constructing the following equation to provide the curve $S(\tau)$. The maximal Lyapunov exponent is the slope of this curve in the scaling region.

$$S(\tau) = \frac{1}{T} \sum_{t=1}^{T} \ln\left(\frac{1}{|U_t|} \sum_{i\in U_t} dist(\chi_t, \chi_i; \tau)\right) \qquad (6)$$

where U_t is the neighborhood set and $dist(\chi_t,\chi_i;\tau)$ defines the distance between a reference trajectory χ_t and a neighbor χ_i after the relative time τ.

When noise is present in the data, the slope of the curve $S(\tau)$ changes as follows:

$$s(\tau) \approx \lambda + \left[\frac{\sigma_{i,\tau}}{dist(x_t, x_i; \tau)}\right]_t - \left[\frac{\sigma_{i,\tau-1}}{dist(x_t, x_i; \tau-1)}\right]_t \qquad (7)$$

λ is the estimate of the maximal Lyapunov exponent and $\sigma_{i,\tau}$ is the standard deviation of the noise. $S(\tau)$ does not contain the embedding dimension explicitly, but nevertheless it enters. This requires that one fix a dimension m for the delay trajectories[2].

3.2 Gençay and Dechert algorithm (1992)

Gençay and Dechert (1992) try to solve the problem in the Lyapunov exponent estimation when a high level of noise is present, by using an algorithm for the estimation of λ, based on feedforward neural networks. We present briefly their estimation procedure below. We notice that for the neural networks estimation we use the method of non-linear least squares (Kuan and Liu, 1995).

In practice it is very difficult to observe the state of the system and know the actual functional form f that generates the dynamics. The model that it is principally used is the following: associated with the dynamical system in equation (3) there is a viewer function $h : R^n \longrightarrow R$ which generates data:

$$x_t = h(y_t) \qquad (8)$$

We suppose that all that is available to the researcher is the sequence of the variables $\{x_t\}$. The well-known Takens' theorem (1981) states that, when $m \geq 2n+1$ we have:

$$J^m(y) = (h(y), h(f(y)), \ldots, h(f^{m-1}(y))) \qquad (9)$$

which is generically an embedding, m the embedding dimension and n the dimension of the real system. For a function $g : R^m \longrightarrow R^m$ for which $J^m \circ f = g \circ J^m$ on an indecomposable attractor, Dechert and Gençay (1990) show that n largest Lyapunov exponents of g are the Lyapunov exponents of f. Thus, they estimate the function g based on the data sequence $\{J^m(y_t)\}$ and calculate the Lyapunov exponents of g.

The mapping g, which is to be estimated may be given as follows:

$$g : \begin{bmatrix} x_{t+m-1} \\ x_{t+m-2} \\ \ldots\ldots \\ \ldots\ldots \\ x_y \end{bmatrix} \rightarrow \begin{bmatrix} u(x_{t+m-1}, x_{t+m-2}, \ldots., x_t) \\ x_{t+m-1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x_{t+1} \end{bmatrix}$$

and this reduces to estimating $x_{t+m} = u(x_{t+m-1}, x_{t+m-2}, \ldots, x_t)$.

Finally, for a single-layer network the least-squares criterion for a data set of length T is:

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2 For more details in the choice of embedding dimension, see Kantz (1994).

$$L(\beta, w, b) = \sum_{t=0}^{T-m-1} [x_{t+m} - u_{N,m}(x_t^m;\ \beta, w, b)]^2 \qquad (10)$$

where: $x_t^m = (x_{t+m-1}, x_{t+m-2}, \ldots, x_t)$ is the input,

$u_{N,m}(x_t^m;\ \beta, w, b)$ is the single-layer feed forward network,

$\varphi(u) = \dfrac{1}{1+\exp(-u)}$ is the activation function,

β,w,b: parameters to be estimated,

N is the number of hidden units.

4. The surrogate data test

The surrogate data test has been proposed by Theiler et al. (1992) and vastly applied to real data. Evidence of non-linearity was often reported while in few works the null hypothesis could not be rejected (Prichard and Price, 1993).

The main idea of this test is to discriminate non-linear dynamics, if this can be detected from the given series. Otherwise the null hypothesis cannot be rejected, which does not necessarily mean that the examined process is stochastic linear. This is only one possible case. There are a number of other possibilities, such as the underlying dynamics is non-linear but masked by noise, or the dimensionality is high and the data size small, so that detection of non-linearity cannot be archived, or simply the data record does not represent well the underlying system.

To test the null hypothesis H_0 that the original signal is generated by a linear stochastic process undergoing a static possibly non-linear transform, an ensemble of M surrogate data sets representing H_0 is generated. To make this, the surrogate data must have the same autocorrelation and the same empirical amplitude distribution as the original signal. Then, a non-linear method is applied to the original and the surrogate data giving the statistics q_0 for the original and $q_1,\ldots,q_M$ for the surrogates. The H_0 is rejected if q_0 is statistically different from $q_1,\ldots,q_M$. Typically, the confidence of rejection is given in terms of the significance S:

$$S = \frac{|q_0 - \overline{q}|}{\sigma_q}$$

where $\overline{q}$ is the average and σ_q the standard deviation of q_i, i=1,…,M.

Significance of about 2σ suggests the rejection of H_0 at the 95% level of confidence. The computation of S quantifies better the difference between original and surrogate data than the simple ordering of the M+1 q-quantities followed in other works (Schreiber, 1999). For the generation of the surrogate data the algorithm of amplitude adjusted Fourier transform (AAFT) is usually applied.

The surrogate date test can be also used as a validation test. After obtaining the surrogate series, we can apply the correlation dimension and the Lyapunov exponents methods. The comparison between the resulting correlation dimensions and Lyapunov exponents (original and surrogate data) can allow us to determine the robustness of the obtained results. For some recent applications of the previous nonlinear tests to financial returns series see Kyrtsou (2002), Kyrtsou and Terraza (2002a,b).

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