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Possajennikov, Alex

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Author: Alex Possajennikov

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The Evolutionary Stability of Constant Consistent Conjectures

Alex Possajennikov*

School of Economics

University of Nottingham

University Park

Nottingham NG7 2RD

Britain

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Abstract

In the general context of smooth two-player games, this paper shows that there is a close connection between (constant) consistent conjectures in a given game and the evolutionary stability of these conjectures. Evolutionarily stable conjectures are consistent and consistent conjectures are the only interior candidates to be evolutionarily stable. Examples are provided to illustrate the result.

Keywords: consistent conjectures, evolutionary stability, indirect evolution

JEL Codes: C72, D43, L13

*Email: alex.possajennikov@nottingham.ac.uk, tel: +44 115 9515461, fax: +44 115 9514159, http://www.nottingham.ac.uk/~lezap.

1 Introduction

Conjectures (or conjectural variations, CV) were introduced in the industrial organization literature to provide a unified framework of imperfect competition, and in this capacity they are still used in empirical investigations (Cabral, 2000, Ch. 9). However, from a theoretical point of view, conjectural variations are considered to be problematic, as a good justification for them is difficult to find in a static setting. For example, it was argued that even consistent conjectures cannot be properly rationalized in such a case (e.g. Lindh, 1992).

Figuieres et al. (2004) survey attempts to justify conjectural variations in a dynamic setting. There are two main dynamic approaches. One considers a repeated game setting with players behaving optimally to various degrees (see e.g. Dockner, 1992; Friedman and Mezzetti, 2002; Jean-Marie and Tidball, 2006). The other approach considers evolutionary models of myopic players. With this approach, in the context of a linear-quadratic duopoly model with differentiated goods, Müller and Normann (2005) showed that consistent conjectures are evolutionarily stable, while Dixon and Somma (2003) demonstrated in their model that an explicit evolutionary process converges to consistent conjectures when goods are homogeneous.

In this paper I show that the consistency of (constant) conjectures and their evolutionary stability are closely connected in the general setting of smooth two-player games. Consistent conjectures are evolutionarily stable not only for linear-quadratic duopoly models, but also for other well-behaved games. Thus this paper provides a rationale for consistent conjectures as they emerge as a stable point of an evolutionary process in many situations. The paper also provides a more convenient way to find stable points of the evolutionary process as consistent conjectures are often easier to find.

The result is illustrated on a set of examples including games with quadratic payoff functions, semi-public good games, and contest games. While the first example is a straightforward generalization of the previously known result, the evolutionary stability of consistent conjectures in the other two examples show how the general connection between the two notions works in other situations.

2 The Model

2.1 The Game

Consider a two-player game $G = (\{1,2\}, \{X_1, X_2\}, \{u_1, u_2\})$, where $X_1, X_2 \subset \mathbb{R}$ are convex strategy spaces and $u_1(x_1, x_2), u_2(x_1, x_2)$ are payoff functions of the two players. In what follows, i refers to either Player 1 or Player 2, and j to the other player ($j \neq i$). The payoff functions are assumed to be twice continuously differentiable.

I consider the following variant of the conjectural variations (CV) model. The players have (constant) conjectures about the marginal reaction of the opponent to a marginal change in strategy. Let $r_{ij} \in Y_i \subset \mathbb{R}$, where Y_i is a convex set, be this conjecture of Player i about Player j, that is, Player i believes that

$$\frac{dx_j}{dx_i}(x_i, x_j) = r_{ij} \; \forall x_i, x_j.$$

I work with constant conjectures because they allow some selection; if conjectures depend on x_i, x_j then many strategy profiles can be supported by (weakly) consistent conjectures (Laitner, 1980; Boyer and Moreaux, 1983).

Since Player i believes that x_j depends on x_i, Player i attempts to maximize $u_i(x_i, x_j(x_i))$ on X_i. At an interior solution x_i^* of this problem $\partial u_i/\partial x_i(x_i^*, x_j(x_i^*)) + \partial u_i/\partial x_i(x_i^*, x_j(x_i^*)) \cdot dx_j/dx_i = 0$. Since the player does not attempt to conjecture the whole reaction function $x_j(x_i)$ but only its slope $dx_j/dx_i = r_{ij}$, x_j is an independent variable, so at x_i^* it holds that

$$\frac{\partial u_i}{\partial x_i}(x_i^*, x_j) + r_{ij} \cdot \frac{\partial u_i}{\partial x_j}(x_i^*, x_j) = 0.$$

When $\partial u_i/\partial x_j(x_i^*, x_j) \neq 0$, then r_{ij} can be equated to a ratio of partial derivatives:

Claim 1 *At an interior best response x_i^* of Player i*

$$r_{ij} = -\frac{\partial u_i/\partial x_i(x_i^*, x_j)}{\partial u_i/\partial x_j(x_i^*, x_j)}, \tag{1}$$

when $\partial u_i/\partial x_j(x_i^*, x_j) \neq 0$.

Condition 1 $\partial u_i/\partial x_j(x_i^*, x_j) \neq 0$ *for all* x_j *and corresponding interior best responses* x_i^*.

Condition 1 is a natural condition in a strategic setting, requiring that a player's payoff depends on the other player's action (at the appropriate best response). If this were not the case, the player would not need to care about the other's response, making conjectures unnecessary.

Let

$$F_i(x_i, x_j; r_{ij}) := \frac{\partial u_i}{\partial x_i}(x_i, x_j) + r_{ij} \cdot \frac{\partial u_i}{\partial x_j}(x_i, x_j).$$

At an interior solution x_i^* of Player i's maximization problem $F_i(x_i^*, x_j; r_{ij}) = 0$. If the solution is unique and interior for each x_j, $F_i(x_i, x_j; r_{ij}) = 0$ implicitly defines the reaction function $x_i^*(x_j; r_{ij})$ of Player i. To be able to use this reaction function I require

Condition 2 *For all* $r_{ij} \in Y_i$*, all* $x_j \in X_j$*, the maximization problem of Player* i *has a unique interior solution* x_i^**, for* $i = 1, 2$.

The condition guarantees that the reaction functions of the players are defined by the equations $F_i(x_i, x_j; r_{ij}) = 0$, $F_j(x_i, x_j; r_{ji}) = 0$.

2.2 Consistent Conjectures

To distinguish the consistency notion I use from that in (some of) the literature (e.g. Bresnahan, 1981, where consistent conjectures are functions all whose derivatives are required to coincide with the corresponding derivatives of the actual reaction function in the neighborhood of equilibrium) I call a conjecture of Player i *weakly consistent* if the conjectured reaction of Player j equals the actual slope of the reaction function of Player j *at best response*, i.e. $r_{ij} = dx_j^*/dx_i(x_i; r_{ji})$ at x_i^*.

When the reaction function is determined implicitly by $F_j(x_i, x_j; r_{ji}) = 0$, and $\partial F_j/\partial x_j(x_i, x_j) \neq 0$, the slope of the reaction function can be found from the implicit function theorem. Therefore,

Claim 2 *At the best responses* (x_i^*, x_j^*)*, conjecture* r_{ij} *is weakly consistent iff*

$$r_{ij} = \frac{dx_j^*}{dx_i}(x_i^*; r_{ji}) = -\frac{\partial F_j/\partial x_i(x_j^*, x_i^*, r_{ji})}{\partial F_j/\partial x_j(x_j^*, x_i^*; r_{ji})}, \tag{2}$$

when $\partial F_j/\partial x_j(x_i^*, x_j^*; r_{ji}) \neq 0$.

Condition 3 $\partial F_j/\partial x_j(x_i^*, x_j^*; r_{ji}) \neq 0$ *for all* $r_{ij} \in Y_i$, $r_{ji} \in Y_j$.

Condition 3 implies that the reaction function is continuous and differentiable at the mutual best response point, and its slope is well defined. The conjecture can then be compared with the slope.

Conjectures r_{ij}^C, r_{ji}^C are mutually consistent if $r_{ij}^C = dx_j^*/dx_i(x_i^*; r_{ji}^C)$ and $r_{ji}^C = dx_i^*/dx_j(x_j^*; r_{ij}^C)$. When the game is symmetric, the reaction functions are symmetric. Then a symmetric conjecture $r^C = r_{ij}^C = r_{ji}^C$ is consistent when $r^C = dx_i^*/dx_j(x_j^*; r^C)$.

2.3 The Evolutionary Stability of Conjectures

Suppose that conjectures are something players are born with (one can interpret them as e.g. optimism/pessimism attitudes). Consider two large populations of players who are repeatedly randomly matched. There is a certain distribution of conjectures in the populations. In a match, players either observe each other's conjectures and behave according to an equilibrium of the game with these conjectures, or they learn to play an equilibrium, where learning is (much) faster than the evolution of conjectures. In either case, the (evolutionary) success of a given conjecture is determined by averaging the equilibrium payoffs of the players endowed with this conjecture over all matches. The proportions of players with given conjectures change according to their evolutionary success.

For a conjecture r_{ji} of Player j, the *evolutionarily stable* (ES) conjecture of Player i is a conjecture r_{ij}^{ES} such that no other conjecture r_{ij} performs better or equally well in a population of Players i almost exclusively composed of players with conjecture r_{ij}^{ES} (and the rest of the population has conjecture r_{ij}). If in a monomorphic population of players with conjecture r_{ij}^{ES} a small proportion of mutants with some other conjecture r_{ij} appears, evolutionary forces will eliminate the mutants.

The informal description in the previous paragraphs corresponds to a generalization of the indirect evolution approach of Güth and Yaari (1992) to asymmetric games and multiple equilibria. More formally, let $u_i(r_{ij}, r_{ji}) =$

$u_i(x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji}))$ be the payoff of Player i when for each pair of conjectures r_{ij}, r_{ji} a particular equilibrium $x_i^*(r_{ij}, r_{ji})$, $x_j^*(r_{ij}, r_{ji})$ is played. For a conjecture r_{ji} of Player j, conjecture r_{ij}^{ES} of Player i is evolutionarily stable under equilibrium selection $x_i^*(r_{ij}, r_{ji})$, $x_j^*(r_{ij}, r_{ji})$ if $u_i(r_{ij}^{ES}, r_{ji}) > u_i(r_{ij}, r_{ji})$ for any $r_{ij} \neq r_{ij}^{ES}$ (asymmetric games ESS; Selten, 1980). A conjecture of Player i is evolutionarily stable against a given conjecture r_{ji} of Player j if it is the unique best response to this conjecture r_{ji} in the game with payoffs $u_i(r_{ij}, r_{ji})$.

If the solutions of their optimization problems are interior, x_i^*, x_j^* satisfy $F_i(x_i^*, x_j^*; r_{ij}) = 0$ and $F_j(x_i^*, x_j^*; r_{ji}) = 0$. Consider the problem

$$\begin{aligned} \max_{x_i, x_j, r_{ij}} \ & u_i(x_i, x_j) \\ \text{s.t.} \ F_i(x_i, x_j; r_{ij}) &= 0 \\ F_j(x_i, x_j; r_{ji}) &= 0 \end{aligned} \tag{3}$$

By the implicit function theorem, the system of equations $F_i(x_i, x_j; r_{ij}) = 0$, $F_j(x_i, x_j; r_{ji}) = 0$ determines locally functions $x_i^*(r_{ij}, r_{ji})$, $x_j^*(r_{ij}, r_{ji})$ when $\partial F_i/\partial x_i \cdot \partial F_j/\partial x_j - \partial F_i/\partial x_j \cdot \partial F_j/\partial x_i \neq 0$ at $r_{ij}, r_{ji}, x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$. Substituting the functions, problem (3) is equivalent to

$$\max_{r_{ij}} u_i(x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})) \tag{4}$$

Problem (4) is exactly the problem of finding a best response conjecture for Player i against the conjecture r_{ji} of Player j.

Since problems (3) and (4) are equivalent, they have the same solution. At an interior solution of problem (3) the following first order conditions hold:

$$\frac{\partial u_i}{\partial x_i} + \lambda \frac{\partial F_i}{\partial x_i} + \mu \frac{\partial F_j}{\partial x_i} = 0 \tag{5a}$$

$$\frac{\partial u_i}{\partial x_j} + \lambda \frac{\partial F_i}{\partial x_j} + \mu \frac{\partial F_j}{\partial x_j} = 0 \tag{5b}$$

$$\frac{\partial u_i}{\partial r_{ij}} + \lambda \frac{\partial F_i}{\partial r_{ij}} + \mu \frac{\partial F_j}{\partial r_{ij}} = 0 \tag{5c}$$

where λ, μ are Lagrangean multipliers. Since u_i does not depend directly on r_{ij}, $\partial u_i/\partial r_{ij} = 0$. Since F_j does not depend directly on r_{ij}, $\partial F_j/\partial r_{ij} = 0$.

Furthermore, since $F_i = \partial u_i/\partial x_i + r_{ij} \cdot \partial u_i/\partial x_j$, $\partial F_i/\partial r_{ij} = \partial u_i/\partial x_j$. By Condition 1 $\partial u_i/\partial x_j \neq 0$, thus $\partial F_i/\partial r_{ij} \neq 0$. From (5c) $\lambda = 0$ and from (5b) $\mu \neq 0$ and $\partial F_j/\partial x_j \neq 0$. Then (5a) and (5b) become $\partial u_i/\partial x_i + \mu \cdot \partial F_j/\partial x_i = 0$ and $\partial u_i/\partial x_j + \mu \cdot \partial F_j/\partial x_j = 0$. Therefore,

Claim 3 *At an interior solution of Problem (4)*

$$\frac{\partial u_i/\partial x_i}{\partial u_i/\partial x_j} = \frac{\partial F_j/\partial x_i}{\partial F_j/\partial x_j}. \tag{6}$$

Condition 4 *For all $r_{ij} \in Y_i, r_{ji} \in Y_j$, at $r_{ij}, r_{ji}, x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$*

$$\frac{\partial F_i}{\partial x_i}\frac{\partial F_j}{\partial x_j} - \frac{\partial F_i}{\partial x_j}\frac{\partial F_j}{\partial x_i} \neq 0.$$

Condition 4 rules out degenerate cases such as when the reaction functions are tangent to each other thus small changes in r_{ij}, r_{ji} may lead to a jump to another equilibrium or to multiple neighboring equilibria, in which cases functions $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$ would be discontinuous or ill-defined. The condition makes sure that there is a smooth selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$ from the set of equilibria as r_{ij}, r_{ji} vary in their respective Y_i, Y_j.[1]

Using Claims 1 and 3, if an interior conjecture r_{ij} is evolutionarily stable, then

$$r_{ij} = -\frac{\partial u_i/\partial x_i}{\partial u_i/\partial x_j} = -\frac{\partial F_j/\partial x_j}{\partial F_j/\partial x_i}.$$

By Claim 2 this means that r_{ij} is weakly consistent. Thus

Proposition 1 *Suppose Conditions 1 to 4 are satisfied. If an interior conjecture r_{ij} is evolutionarily stable against r_{ji} under some smooth equilibrium selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$ then it is weakly consistent for this r_{ji}.*

The proposition may also be stated as follows:

Corollary 1 *Suppose Conditions 1 to 4 are satisfied. If an interior conjecture r_{ij} is not weakly consistent for r_{ji}, then it is not evolutionarily stable against r_{ji} under any smooth equilibrium selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$.*

[1] The problem of equilibrium selection disappears if there is a unique equilibrium for all r_{ij}, r_{ji}. This will be the case in the examples in Section 3.

For clarity, the statements are formulated using all four conditions discussed above. Not all of them are independent, and they are sufficient for the statements but not all are necessary. Local versions of conditions may be sufficient for the result; e.g. Condition 1 needs to hold only at the best response corresponding to evolutionarily stable r_{ij}. From the proof of Claim 3, Condition 3 is implied by Conditions 1 and 2 at an interior evolutionarily stable r_{ij}, thus a local version of Condition 3 is automatically satisfied. An equilibrium selection may exist even when $\partial F_i/\partial x_i \cdot \partial F_j/\partial x_j - \partial F_i/\partial x_j \cdot \partial F_j/\partial x_i = 0$, and not all best responses need to be interior. The present formulation is chosen to avoid excessive technicalities, and in the examples in Section 3 the conditions will be satisfied globally.

Additional assumptions are needed for the inference from consistency to evolutionary stability. Condition 4 guarantees the existence of equilibrium selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$. If sufficient conditions for a unique global optimum of Problem (4) are satisfied, then the weak consistency of an interior conjecture r_{ij}^* implies that r_{ij}^* is an ES conjecture. One such sufficient condition is the global concavity of the payoff function.

Proposition 2 *Suppose Conditions 1 to 4 are satisfied. If an interior conjecture r_{ij}^* is weakly consistent for r_{ji} and $[u_i(x_i^*(r, r_{ji}), x_j^*(r, r_{ji}))]_r'' < 0$ for all r, then r_{ij}^* is evolutionarily stable against r_{ji} under the equilibrium selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$.*

Another condition, often easier to check, is local concavity together with the uniqueness of the critical point. Thus

Proposition 3 *Suppose Conditions 1 to 4 are satisfied. If an interior conjecture r_{ij}^* is weakly consistent, $[u_i(x_i^*(r, r_{ji}), x_j^*(r, r_{ji}))]_r' = 0$ has a unique solution r_{ij}^*, and $[u_i(x_i^*(r, r_{ji}), x_j^*(r, r_{ji}))]_r''|_{r=r_{ij}^*} < 0$, then r_{ij}^* is evolutionarily stable under the equilibrium selection $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$.*

The analysis above is for Player i and for a conjecture r_{ji} of Player j. A similar analysis can be performed for Player j, keeping constant a conjecture r_{ij} of Player i. Then if interior conjectures r_{ij}^*, r_{ji}^* are mutually evolutionarily stable, they are mutually consistent. If interior conjectures are not mutually consistent, then they are not mutually evolutionarily stable. Analogous extensions hold for the other propositions.

When the game is symmetric, it is natural to expect players to hold symmetric conjectures. In the symmetric case, the matching can be done within one population since players' roles are indistinguishable. Although in this case evolutionary stability is not equivalent to strict best response (strict best response implies evolutionary stability but not the reverse), the propositions hold for the one-population symmetric case as well. An interior evolutionarily stable conjecture is a best response to itself, so the first order conditions have to hold, thus implying Proposition 1 and Corollary 1. Since a strict symmetric equilibrium is evolutionarily stable in the one-population symmetric case, sufficient conditions of Propositions 2 and 3 imply evolutionary stability in this case too.

A graphical illustration of the close connection between consistency and evolutionary stability is given in an example in the next section. Intuitively, r_{ji} determines the reaction function of Player j. By varying r_{ij}, Player i can change his own reaction function and so can change its point of intersection with the reaction function of Player j. Player i will choose a point on the reaction function of Player j where it is tangent to a level curve of Player i's payoff function. But since r_{ij} by Claim 1 equals the slope of this level curve, best response r_{ij} has to be equal to the slope of the reaction function of Player j, i.e. be consistent.

If conjecture r_{ij} is consistent, Player i "knows" the reaction of Player j to small changes in x_i. Thus a player with consistent conjecture maximizes the "correct" function $u_i(x_i, x_j(x_i))$, and so has higher payoff than when the conjecture is not consistent. Therefore the obtained result may look obvious. Nevertheless, Müller and Normann (2005, p. 500) state that "[...] the result that the evolutionarily stable conjectures coincide with the consistent conjectures is surprising as there is no obvious analogy between the two concepts." The result was also surprising for me. There was no reason to expect apriori that 'more rationality' (consistency) should lead to the same result as 'less rationality' (evolution); only after interpreting the result did the connection appear obvious.

3 Examples

3.1 Games with Quadratic Payoff Function

Consider the class of symmetric games with the payoff function given by

$$u_i(x_i, x_j) = Ax_i + Bx_ix_j - Cx_i^2,$$

where A, B, C are parameters, $A > 0$, $B \neq 0$, $C > 0$, and $C > |B|$. Several well-known games fall in this class, among which are the games for which the evolutionary stability of consistent conjectures was analyzed in the literature.

For example, in the differentiated goods Cournot duopoly with linear inverse demands $P_i(q_i, q_j) = a - b_iq_i - b_jq_j$ and quadratic costs $C_i(q_i) = cq_i^2$, analyzed in Müller and Norman (2005) (and in Dixon and Somma, 2003, for the homogenous goods case), a firm's profit $\pi_i(q_i, q_j) = P_i(q_i, q_j)q_i - C_i(q_i)$ can be represented as $\pi_i(q_i, q_j) = aq_i - b_jq_iq_j - (b_i + c)q_i^2$. A variant of the search model in Milgrom and Roberts (1990) leads to payoff function $u_i(x_i, x_j) = Ax_i + \alpha x_ix_j - cx_i^2$, where the gains from trade $Ax_i + \alpha x_ix_j$ depend on the efforts x_i, x_j of the players and the cost of effort is cx_i^2.

Let the strategy space be $X = [0, \overline{x})$, where $\overline{x}$ is a suitable upper bound to make economic sense. There may be no upper bound ($\overline{x} = \infty$). Let the conjecture space be $Y = (-1, 1)$.

Since $\partial u_i/\partial x_j = Bx_i \neq 0$ in the interior of X, Condition 1 is satisfied. Player i's problem is to maximize $Ax_i + Bx_ix_j - Cx_i^2$, when $dx_j/dx_i = r_{ij}$. The first order condition is

$$F_i = A + Bx_j + Br_{ij}x_i - 2Cx_i = 0.$$

This implies $x_i^* = (A + Bx_j)/(2C - Br_{ij})$. If $B > 0$, then consider $\overline{x} = \infty$ for x_i^* to be interior. If $B < 0$, then $\overline{x} = -A/B$ guarantees that x_i^* is interior. Since $\partial F_i/\partial x_i = Br_{ij} - 2C$ and $\partial F_i/\partial x_j = B$, the second order condition for Player i's maximization problem is $\partial F_i/\partial x_i + r_{ij} \cdot \partial F_i/\partial x_j = 2Br_{ij} - 2C < 0$. Thus Condition 2 is satisfied and the reaction functions are given by $F_i = 0$, $F_j = 0$.

Since $\partial F_i/\partial x_i = Br_{ij} - 2C \neq 0$, Condition 3 is satisfied. Finally, consider $\partial F_i/\partial x_i \cdot \partial F_j/\partial x_j - \partial F_i/\partial x_j \cdot \partial F_j/\partial x_i = (Br_{ij} - 2C)(Br_{ji} - 2C) - B^2$. Since

$C > |B|$ and $r_{ij}, r_{ji} \in (-1,1)$, then $Br_{ij} - 2C < -C$ and $Br_{ji} - 2C < -C$. Since the terms in the parentheses are negative, the right-hand side expression is larger than $(-C)(-C) - B^2 > 0$. Condition 4 is also satisfied.

The consistent symmetric conjecture can be found from

$$r = -\frac{\partial F_i/\partial x_j}{\partial F_i/\partial x_i} = -\frac{B}{Br - 2C}.$$

Then $Br^2 - 2Cr + B = 0$. When $B < 0$, then there is one root on $(-1,1)$ and it is between -1 and 0. When $B > 0$, then the root is between 0 and 1. In any case, there is a unique consistent conjecture $r^C \in (-1,1)$. By Corollary 1 it is the unique interior candidate for an evolutionarily stable conjecture.

The payoff function can be written as $u_i(x_i, x_j) = x_i^2\left((A + Bx_j)/x_i - C\right)$. From the reaction functions, $A + Bx_j^* = (2C - Br_{ij})x_i^*$. Therefore, at equilibrium $\partial u_i/\partial r_{ij} = -B\left(x_i^*\right)^2 + 2x_i^*\left(C - Br_{ij}\right) \cdot \partial x_i^*/\partial r_{ij}$.

The equilibrium strategy of Player i is

$$x_i^* = \frac{A(B - (Br_{ji} - 2C))}{\left(Br_{ij} - 2C\right)\left(Br_{ji} - 2C\right) - B^2}.$$

Then $\partial x_i^*/\partial r_{ij} = \left(-B\left(Br_{ji} - 2C\right)x_i^*\right)/\left(\left(Br_{ij} - 2C\right)\left(Br_{ji} - 2C\right) - B^2\right)$, and

$$\frac{\partial u_i}{\partial r_{ij}} = \frac{B^2\left(x_i^*\right)^2}{\left(Br_{ij} - 2C\right)\left(Br_{ji} - 2C\right) - B^2}(B - r_{ij}(2C - Br_{ji})).$$

The unique solution of $\partial u_i/\partial r_{ij} = 0$ is $r_{ij} = -B/(Br_{ji} - 2C)$. When $r_{ji} = r^C$, the unique solution is $r_{ij} = r^C$.

Furthermore,

$$\frac{\partial^2 u_i}{\partial r_{ij}^2} = B^2\left(x_i^*\right)^2 \frac{Br_{ji} - 2C}{\left(Br_{ij} - 2C\right)\left(Br_{ji} - 2C\right) - B^2}$$

at $r_{ij} = r_{ji} = r^C$. The denominator of this expression is positive, and the numerator is negative, for all $r_{ij}, r_{ji} \in (-1,1)$. Thus $\partial^2 u_i/\partial r_{ij}^2 < 0$ and by Proposition 3

Proposition 4 *In the games with a quadratic payoff function analyzed in this section, there exists a unique consistent conjecture and it is the unique evolutionarily stable one.*

To get the intuition behind the result, also for the general case of the previous section, consider Figure 1.

[Figure 1 around here]

Conjecture r_{21} determines a reaction function of Player 2, which is linear and decreasing when $B < 0$. Varying r_{12} varies the reaction function of Player 1. Three of these reaction functions are drawn in the figure. The equilibrium is on the intersection of the reaction functions, thus varying r_{12} allows Player 1 to move along the given reaction function of Player 2. Some payoff level curves of Player 1 are also drawn in the figure. Payoff is increasing in the south-east direction.

Since Player 1 can vary the equilibrium point by moving along the reaction function of Player 2, the best payoff Player 1 can achieve in equilibrium is at the point where a level curve is tangent to the reaction function of Player 2. At this point the slope of the reaction function equals the slope of the payoff level curve. For Player 1, having a conjecture r_{12} means that the reaction function of Player 1 cuts the level curves at points where their slope equals r_{12} (by Claim 1). Therefore, conjecture r_{12} equals the slope of the payoff level curves at the points of intersection with the corresponding reaction function of Player 1. At the best equilibrium for Player 1, a level curve and the reaction function of Player 2 are tangent and so evolutionary stable conjecture r_{12} is equal to the slope of the reaction function of Player 2, which means that r_{12} is consistent.

3.2 Semi-Public Good Games

Consider the following symmetric two player public good provision game. Players have endowments w of private good. They can contribute x_i to the public good, and leave $y_i = w - x_i$ of private good for consumption. Let the strategy set be $X = [-w, w]$, which is needed to guarantee an interior best response and can be interpreted as opportunities to contribute as well as to take out of a common pool of public good. The contribution of Player j enters Player i's utility with weight $\beta \in (0, 1)$, thus for Player i the total supply of public good is $X_i = x_i + \beta x_j$. Players' utility functions are $u_i(y_i, X_i)$.

This is the model of semi-public goods considered in Costrell (1991). Let $Y = (-1, 1)$.

Suppose that the utility functions are $u_i(y_i, X_i) = y_i^\alpha X_i^{1-\alpha}$, where $0 < \alpha < 1$.[2] The payoff function of Player i is then

$$u_i(x_i, x_j) = (w - x_i)^\alpha (x_i + \beta x_j)^{1-\alpha}.$$

For a conjecture r_{ij}, the first order condition of the maximization problem of Player i is

$$F_i = -\alpha \left(\frac{X_i}{y_i}\right)^{1-\alpha} + (1-\alpha)\left(\frac{y_i}{X_i}\right)^\alpha (1 + \beta r_{ij}) = 0.$$

Let $v_i = X_i/y_i$. Then the first order condition implies that $v_i = (1+\beta r_{ij})(1-\alpha)/\alpha$. This means that $x_i + \beta x_j = (1 + \beta r_{ij})(w - x_i)(1 - \alpha)/\alpha$, or that

$$x_i^* = \frac{(1-\alpha)(1+\beta r_{ij})}{1 + (1-\alpha)\beta r_{ij}} w - \frac{\alpha\beta}{1 + (1-\alpha)\beta r_{ij}} x_j.$$

This x_i^* is unique and interior for all $x_j \in [-w, w]$. Note also that $X_i^* = x_i^* + \beta x_j = (1 + \beta r_{ij})(w - x_i^*)(1 - \alpha)/\alpha > 0$ and $y_i^* = w - x_i^* > 0$ at the interior best response.

Because

$$\begin{aligned}
\frac{\partial F_i}{\partial x_i} &= -\alpha(1-\alpha)(y_i + X_i)\left(\left(\frac{X_i}{y_i}\right)^{-\alpha}\frac{1}{y_i^2} + \left(\frac{y_i}{X_i}\right)^{\alpha-1}\frac{1}{X_i^2}(1+\beta r_{ij})\right) \\
\frac{\partial F_i}{\partial x_j} &= -\alpha(1-\alpha)\beta y_i\left(\left(\frac{X_i}{y_i}\right)^{-\alpha}\frac{1}{y_i^2} + \left(\frac{y_i}{X_i}\right)^{\alpha-1}\frac{1}{X_i^2}(1+\beta r_{ij})\right),
\end{aligned}$$

it holds that $\partial F_i/\partial x_i + r_{ij} \cdot \partial F_i/\partial x_j = -\alpha(1-\alpha)(X_i + y_i(1+\beta r_{ij}))(v_i^{-\alpha} \cdot 1/y_i^2 + v_i^{1-\alpha}(1+\beta r_{ij})/X_i^2) < 0$ at x_i^*, since $X_i > 0$, $y_i > 0$, and $1 + \beta r_{ij} > 0$ when $r_{ij} \in (-1, 1)$. This means that Condition 2 is fulfilled, and reaction functions are given by $F_i = 0$, $F_j = 0$.

From the reaction function, with a consistent symmetric conjecture

$$r = \frac{dx_i^*}{dx_j} = -\frac{\beta\alpha}{1 + (1-\alpha)\beta r},$$

[2] To cover the possibility of a negative X_i, which does not arise in equilibrium, assume $u_i(y_i, X_i) = 0$ if $X_i < 0$.

implying $(1-\alpha)\beta r^2+r+\alpha\beta = 0$. For $r \in (-1,1)$, there is one root r^C of this equation, and it is between -1 and 0. The consistent conjecture is negative, as obtained in Costrell (1991) for a more general semi-public good setup.

It holds that $\partial u_i/\partial x_j = (1-\alpha)\beta\,(y_i/X_i)^\alpha \neq 0$ in the interior of X, as required by Condition 1. Also $\partial F_i/\partial x_i \neq 0$ in equilibrium, thus Condition 3 is fulfilled. Finally, for Condition 4 $\partial F_i/\partial x_i \cdot \partial F_j/\partial x_j - \partial F_i/\partial x_j \cdot \partial F_j/\partial x_i = \alpha^2(1-\alpha)^2(X_i+(1+\beta r_{ij})y_i)(X_j+(1+\beta r_{ji})y_j)(X_iX_j+X_iy_j+y_iX_j+(1-\beta^2)y_iy_j)/((X_iX_j)^{\alpha+1}(y_iy_j)^{2-\alpha}) \neq 0$. By Corollary 1 the consistent conjecture is the unique interior candidate for an evolutionarily stable conjecture.

Since in equilibrium $X_i^* = (1+\beta r_{ij})y_i^*(1-\alpha)/\alpha$, the utility function can be written as $u_i(r_{ij},r_{ji}) = (1+\beta r_{ij})^{1-\alpha}y_i^*\left((1-\alpha)/\alpha\right)^{1-\alpha}$. Then $\partial u_i/\partial r_{ij} = \left((1-\alpha)/\alpha\right)^{1-\alpha}(1+\beta r_{ij})^{-\alpha}\left((1-\alpha)\beta y_i^* + (1+\beta r_{ij})\partial y_i^*/\partial r_{ij}\right)$. The equilibrium strategy of Player i is

$$x_i^* = \frac{(1-\alpha)\left((1+\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - (1+\beta r_{ji})\alpha\beta\right)}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - \alpha^2\beta^2}w.$$

It holds that $\partial y_i^*/\partial r_{ij} = -\partial x_i^*/\partial r_{ij} = -(\beta(1-\alpha)(1+(1-\alpha)\beta r_{ji}))y_i^*/((1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - \alpha^2\beta^2)$. Then

$$\frac{\partial u_i}{\partial r_{ij}} = -\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\frac{(1+\beta r_{ij})^{-\alpha}(1-\alpha)\alpha\beta^2 y_i^*(r_{ij}(1+(1-\alpha)\beta r_{ji})+\alpha\beta)}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - \alpha^2\beta^2}.$$

The first order maximization condition $\partial u_i/\partial r_{ij} = 0$ has the unique solution $r_{ij} = -\alpha\beta/(1+(1-\alpha)\beta r_{ji})$. When $r_{ji} = r^C$, then the solution is $r_{ij} = r^C \in (-1,0)$.

Let $K = -\left((1-\alpha)/\alpha\right)^{1-\alpha}(1-\alpha)\beta^2\alpha < 0$. For the second order condition,

$$\frac{\partial^2 u_i}{\partial r_{ij}^2} = K(1+\beta r_{ij})^{-\alpha}y_i^*\frac{1+(1-\alpha)\beta r^C}{(1+(1-\alpha)\beta r^C)^2 - \alpha^2\beta^2}$$

at $r_{ij} = r_{ji} = r^C$. The sign of $\partial^2 u_i/\partial r_{ij}^2$ is determined by the signs of $1+(1-\alpha)\beta r^C$ and $(1+(1-\alpha)\beta r^C)^2-\alpha^2\beta^2$. Since $1+(1-\alpha)\beta r^C = -\beta\alpha/r^C$, $1+(1-\alpha)\beta r^C > 0$ and $(1+(1-\alpha)\beta r^C)^2-\alpha^2\beta^2 = \alpha^2\beta^2\left(1/(r^C)^2-1\right) > 0$. Therefore $\partial^2 u_i/\partial r_{ij}^2 < 0$, and the consistent conjecture r^C is also evolutionarily stable.

Proposition 5 *In the semi-public good games of this section the unique consistent conjecture r^C is the unique evolutionarily stable conjecture.*

3.3 Rent-seeking games

Consider the following symmetric game, presented as a rent-seeking contest first in Tullock (1980). Two players compete for a prize of value V by making investments x_i. The probability that a player wins the prize is $x_i/(x_i + x_j)$. The cost of an investment is simply x_i. The (expected) payoff of Player i is

$$u_i(x_i, x_j) = \frac{x_i}{x_i + x_j} V - x_i.$$

To avoid technical difficulties that do not influence the result, consider investments strictly between 0 and V, i.e. $X = (0, V)$. Let also $Y = (-1, 1)$.

Since $\partial u_i/\partial x_j = -x_i V/(x_i + x_j)^2$, Condition 1 is satisfied in the interior of X. Note also that $\partial u_i/\partial x_i = x_j V/(x_i + x_j)^2 - 1$.

The first order condition of Player i's optimization problem is

$$F_i = \frac{x_j - r_{ij} x_i}{(x_i + x_j)^2} V - 1 = 0.$$

The solution of this equation satisfies $(x_j - r_{ij} x_i) V = (x_i + x_j)^2$. When $x_i = 0$, then the left-hand side $x_j V$ is larger than the right-hand side x_j^2. When $x_i = V$, then the left-hand side $(x_j - r_{ij} V) V$ is smaller than the right-hand side $(V + x_j)^2$. Since the equation is quadratic, there is a unique solution of the first order condition equation on $(0, V)$ for any $r_{ij} \in (-1, 1)$ and $x_j \in (0, V)$.

It holds that $\partial F_i/\partial x_i = V(-r_{ij}(x_i + x_j) - 2(x_j - r_{ij} x_i))/(x_i + x_j)^3$ and $\partial F_i/\partial x_j = V((x_i + x_j) - 2(x_j - r_{ij} x_i))/(x_i + x_j)^3$. At the solution of the first order condition equation

$$\begin{aligned}
\frac{\partial F_i}{\partial x_i} &= \frac{1}{(x_i + x_j)^2}(-r_{ij} V - 2(x_i + x_j)) \\
\frac{\partial F_i}{\partial x_j} &= \frac{1}{(x_i + x_j)^2}(V - 2(x_i + x_j)).
\end{aligned}$$

Then $\partial F_i/\partial x_i + r_{ij} \cdot \partial F_i/\partial x_j = -2(1 + r_{ij})/(x_i + x_j) < 0$. Thus locally the second order condition of Player i's optimization problem is satisfied, and the best response is found from $F_i = 0$, that is, Condition 2 is satisfied.

When $\partial F_i/\partial x_i = 0$ at the solution of the first order condition equation, then $x_i + x_j = -V r_{ij}/2$, or $r_{ij} x_i = -V r_{ij}/2 - r_{ij} x_j$. From the first order condition $r_{ij}x_i = x_j - Vr_{ij}^2/4$. This implies $(r_{ij}+1)x_j = -Vr_{ij}^2/4$. However, this can hold only when $x_j < 0$, thus $\partial F_i/\partial x_i = 0$ and $F_i = 0$ are incompatible on $(0, V)$. Therefore Condition 3 holds.

At the solution of the first order condition equations for the two players $\partial F_i/\partial x_i \cdot \partial F_j/\partial x_j - \partial F_i/\partial x_j \cdot \partial F_j/\partial x_i = V((r_{ij}r_{ji}-1)V + 2(r_{ji}+r_{ij}+2)(x_i + x_j))/(x_i+x_j)^4$. Adding up the first order condition $(x_j - r_{ij}x_i)V/(x_i + x_j)^2 - 1 = 0$ for Player i multiplied by r_{ji} and the first order condition $(x_i - r_{ji}x_j)V/(x_i+x_j)^2 - 1 = 0$ for Player j gives $x_i(1 - r_{ji}r_{ij})V/(x_i + x_j)^2 - r_{ij} - 1 = 0$. Doing the analogous operation interchanging players gives $x_j(1-r_{ij}r_{ji})V/(x_i+x_j)^2 - 1 - r_{ij} = 0$. Adding up, $(1-r_{ij}r_{ji})V/(x_i+x_j) - 2 - r_{ij} - r_{ij} = 0$. Thus $x_i + x_j = (1-r_{ij}r_{ji})V/(2+r_{ij}+r_{ij})$ in equilibrium. But then $(r_{ij}r_{ji}-1)\,V + 2(x_i+x_j)(r_{ji}+r_{ij}+2) = (1-r_{ij}r_{ji})\,V > 0$. Therefore Condition 4 is satisfied.

In equilibrium, the slope of the reaction function is

$$\frac{dx_i^*}{dx_j} = -\frac{\partial F_i/\partial x_j}{\partial F_i/\partial x_i} = -\frac{V - 2(x_i+x_j)}{-Vr_{ij} - 2(x_i+x_j)}.$$

With symmetric conjectures $r_{ij} = r_{ji} = r$, the equilibrium $x_i^* = x_j^* = x$ is symmetric, thus for a consistent symmetric conjecture $r = -(V-4x)/(-Vr-4x)$. In symmetric equilibrium $x = (1-r)V/4$. Then for a conjecture r to be consistent, $r = -(V-(1-r)V)/(-Vr-(1-r)V) = r$. That is, every $r \in (-1, 1)$ is a symmetric consistent conjecture. This conforms to the result of Michaels (1989) who obtains that every r is a consistent conjecture for more general symmetric contests.

The equilibrium strategies of the players are

$$x_i^* = \frac{(1-r_{ij}r_{ji})\,(1+r_{ji})}{(2+r_{ij}+r_{ij})^2}V \text{ and } x_j^* = \frac{(1-r_{ij}r_{ji})\,(1+r_{ij})}{(2+r_{ij}+r_{ij})^2}V.$$

To maximize the fitness function $u_i(r_{ij}, r_{ji})$ at x_i^*, x_j^* find $\partial u_i/\partial r_{ij} = V((x_jV - (x_i+x_j)^2)(-3r_{ji}^2 - r_{ji}^3 - 2 + r_{ij}r_{ji} - 4r_{ji} + r_{ji}^2 r_{ij}) + (-x_i)\,V(-r_{ji} - 3r_{ij}r_{ji} - r_{ji}^2 - 2r_{ij}r_{ji}^2 - r_{ij}))/((x_i+x_j)^2(2+r_{ij}+r_{ji})^3)$. Since in equilibrium $(x_i+x_j)^2 = (x_j - r_{ij}x_i)\,V$,

$$\frac{\partial u_i}{\partial r_{ij}} = \frac{x_i}{(x_i+x_j)^2}\frac{V^2(1+r_{ji})}{(2+r_{ij}+r_{ji})^3}(r_{ij}^2 r_{ji} - r_{ij}(r_{ji}^2+1) + r_{ji}).$$

Solving $r_{ij}^2 r_{ji} - r_{ij}(r_{ji}^2 + 1) + r_{ji} = 0$ gives $r_{ij} = r_{ji}$ and $r_{ij} = 1/r_{ji}$. The latter solution is not admissible since $r_{ij}, r_{ji} \in (-1, 1)$.

For each $r_{ji} \in (-1, 1)$ there is a unique solution $r_{ij} = r_{ji}$ of the first order condition within the $(-1, 1)$ interval. The second order condition is

$$\frac{\partial^2 u_i}{\partial r_{ij}^2} = V^2(1+r)\frac{x_i}{(x_i+x_j)^2}\frac{(r^2-1)(2+2r)}{(2+2r)^4} < 0,$$

at $r_{ji} = r_{ij} = r$. By Proposition 3 this means that all r are evolutionarily stable.

Proposition 6 *In the rent-seeking game of this section, all conjectures* $r \in (-1, 1)$ *are consistent and evolutionarily stable.*

4 Conclusion

It is shown that the observations of Müller and Normann (2005) and Dixon and Somma (2003) about the evolutionary stability of consistent conjectures for a particular duopoly case extend to other games because they are based on the coincidence of the first order conditions. Apart from the examples considered in this paper, other games to which the results can be applied include common pool resource exploitation games and differentiated product Bertrand duopoly. It should be possible to generalize the results to n-player aggregative games, i.e. games in which payoffs depend on own strategy and on an aggregate of strategies of the other players. In such games a conjecture can be treated as the conjecture about the aggregate reaction of the other players.

The intuition for the evolutionary stability of consistent conjectures is that a player with such a conjecture correctly estimates the response of the other player and thus maximizes the "right" function, outperforming in evolutionary terms players with other conjectures. Though this result may appear obvious *ex-post*, it was not so before the analysis. It is interesting that 'more rational' (consistency) and 'less rational' (evolution) approaches lead to the same outcome in many games.

The contribution of the paper can be seen as twofold. The evolutionary approach can provide a justification for consistent conjectures as emerging from a dynamic process, and the paper shows that this justification holds for many situations. On the other hand, consistent conjectures are often easier to find, simplifying the evolutionary analysis. Depending on the questions asked about a game, one or the other approach can be used, since the two approaches complement each other.

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Figure1

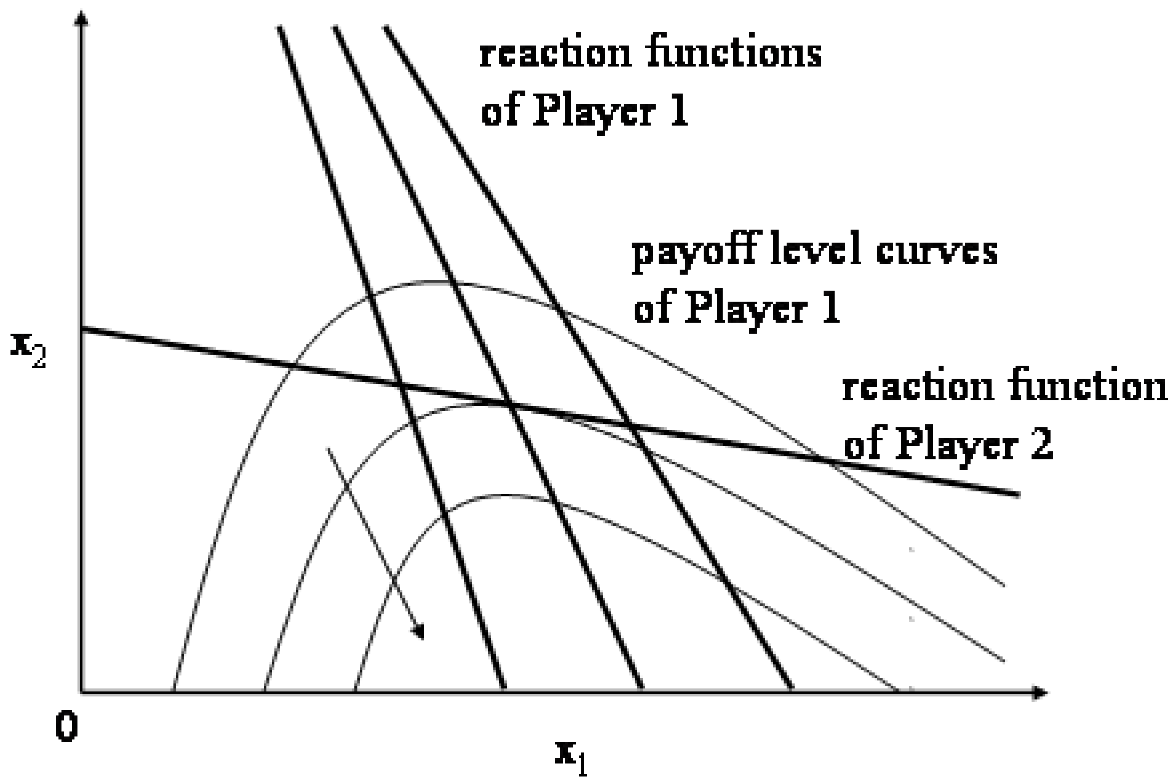

Figure 1: Reaction functions and payoff level curves