

### Land, technical progress and the falling rate of profit

Petith, Howard

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# Land, Technical Progress and the Falling Rate of Profit

Howard Petith<sup>1</sup>

*Universitat Autònoma de Barcelona*  
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## **Abstract**

The paper sets out a one sector growth model with a neoclassical production function in land and a capital-labour aggregate. If the elasticity of substitution between land and the capital-labour aggregate is less than one and if the rate of capital augmenting technical progress is strictly positive, then the rate of profit will fall to zero. This result holds regardless of the rate of land augmenting technical progress: no amount of technical advance in agriculture can stop the fall in the rate of profit. The paper also discusses the relation of this result to the classical and Marxist literature.

*JEL classification:* B24, E11, O41.

*Key words:* Marx, classical economics, the falling rate of profit.

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<sup>1</sup> Universitat Autònoma de Barcelona, Departament d'Economia i d'Història Econòmica, Edifici B, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona) Spain. howard.petith@uab.es, phone 34 93 581 1721, fax 34 93 581 2012. The paper in its present form could hardly have been written without the help of Gerard Duménil, Duncan Foley and Domenic Lévy. It appeared as a UAB working paper in 1992. The following people have made useful comments: Hamid Azari, Jordi Brandts, Roberto Burguet, Ramon Caminal, Simon Emsley, Alan Freeman, John Hamilton and Carmen Matutes. It was also presented at the Bellaterra seminar, the Macro workshop at the UAB, The Atlantic International Economic Conference, El SimposiÓ De Análisi Econòmica, The IWGVT Conference, Nueva Direccions en El Pensamiento Económico, The Conference on New and Old Growth Theories and the III Coloquio de Economistas Políticos de America Latina. The author is responsible for all errors. Finally thanks go to my late wife, Deirdre Herrick, for her considerable help. Financial assistance is acknowledged from the Spanish Ministry of Science and Education and FEDER through grants SEC2003-00306 and SEJ2006-03879, from the Barcelona Economics Program of CREA, from the Generalitat of Catalunya through grant 2005SGR00477, and from Consolider-Ingenio 2010(CSD2006-00016).

## 1. Introduction

This paper is basically about the falling rate of profit. It develops an essentially neoclassical growth model with land, labour and capital as factors of production. Capital accumulates through capitalist savings, the labour supply is infinitely elastic at a subsistence wage and all factors experience factor augmenting technical progress. The result is that if the elasticity of substitution between land and a capital-labour aggregate is less than one and if the rate of technical progress experienced by capital is positive, then the capital-labour ratio rises toward infinity, the share of capital approaches one and the rate of profit falls toward zero. This result holds regardless of the speed of technical progress that land experiences. Surprisingly, technical advance in agriculture cannot halt the fall of the rate of profit.

This introduction discusses the relation of the result of the paper to the classical and Marxist literature. With respect to the classical literature, the conclusions, but not the logic, of the classical authors are supported against those of modern writers. With respect to the Marxist literature, the falling rate of profit is decoupled from rising wages and a coherent way of linking this concept with Marx's overall view of the future of capitalism is provided. Finally the paradoxical relation between the result of this paper and a particular induced innovation mechanism is underlined.

As a bench mark for the classical case, it is convenient to start with a simplified version of the "corn model" with technical progress in agriculture: the production of corn is constant returns to scale in labour and homogenous land. Capitalists rent land from landlords, paying the marginal product of land after the harvest has been collected and hire labor, paying in advance with their accumulated stock of corn. They save a portion of their profits which becomes zero when the rate of profit reaches its minimum level. The labor force grows only when the wage is above subsistence. At each moment the wage is determined so that the entire stock of corn is used to pay wages. The classical model, when stripped of its frills<sup>2</sup>, corresponds to this corn model. One of the main conclusions of the classical school is that the equilibrium of this model will approach a stationary state with the rate of profit at a minimum, the wage at subsistence and no growth. Now add land augmenting technical progress. With the intuition of the neoclassical growth model, one sees that this model has a steady state in which the rate of profit is above the minimum level, the wage is above subsistence,

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<sup>2</sup> One of the frills is non-homogenous land and rent. This is important for distribution but not relevant for the falling rate of profit and the approach to the stationary state.

and output, labour and the stock of corn grow at the rate of technical progress<sup>3</sup>. That is, once land augmenting technical progress is added to the classical model, its equilibrium does not approach the stationary state.

In the light of this, consider the positions of David Ricardo and John Stuart Mill on the falling rate of profit and the approach to the stationary state in the presence of technical progress. Ricardo (1817, p. 120) stated,

The natural tendency of profits then is to fall; for...the additional quantity of food required is obtained by the sacrifice of more and more labour. This tendency...is happily checked at repeated intervals by improvements in machinery connected with the production of necessaries, as well as by discoveries in the science of agriculture...which enable us to lower the price of the prime necessaries of the labourer. But the rise ...in the wages is, however, limited; for as soon as wages should be equal...to...the whole receipts of the farmer, there must be an end to accumulation;...

The common interpretation of this has been that technical progress will only slow the fall of the rate of profit. For example Eltis (1988, p. 278), in the *New Palgrave*, writes of Ricardo that technical progress "...reduces the rate at which profits decline, without affecting the proposition that they must fall eventually to the minimum stationary level." Mill (1965, p. 743) also considered the same issues. He concluded

All improvements, therefore, in production of almost any commodity, tend to widen the interval which has to be passed before arriving at the stationary state.

Again the common interpretation is as with Ricardo. According to Eltis (1988, p. 279) "...Mill did not envisage that technical progress...would be sufficient to overcome the influence of population growth and agricultural diminishing returns so profits would continually fall towards (the minimum level)". Thus, if one takes the corn model as the basis for classical thinking, it must be concluded that Ricardo and Mill did not understand that if there is any technical progress at all, the economy will never arrive at the stationary state and the rate of profit and the wage will be forever above their minimum levels. This lack of understanding, to my knowledge, has not been pointed out before.

Now turn to the modern treatments of technical progress in the context of the classical model: Johansen (1967) and Samuelson (1976). Both of these authors have the classical labour markets and capitalist behavior. Their models differ mainly in production since both have

<sup>3</sup> This is confirmed by the work of modern authors cited below

neoclassical capital rather than corn as an argument in their production functions. This is important because it allows for the possibility of capital augmenting technical progress. Specifically Johansen has a Cobb-Douglas production function in a capital-labor aggregate and land and capital augmenting technical progress while Samuelson has a general neoclassical production function in the same arguments and land augmenting technical progress. Both authors show that their models exhibit steady states with the rate of profit above the minimum level, the wage above subsistence and positive growth. They then state that their results corroborate those of the classical authors. This gives rise to two questions. First, how can these results corroborate those of the classical authors when they are exactly the contrary? And second, how can these models, which are similar to the present model, have a steady state with the rate of profit above its minimum level?

In respect to the first question Johansen's justification (p. 21) is that

In the classical writings one can find some suggestions about technical progress postponing stagnation, perhaps for an infinite future.

Samuelson (p. 1416) notes,

Mill went on to emphasize that technical innovation, continued in the long-run steady state, would imply rising output forever, we can show on Mill's behalf that, if there is land augmenting technical progress at a steady exponential rate (the above described stationary state will occur).

No references for these justifications are given and, in any case, they do not hold water. With respect to Johansen, it might be the case that there are suggestions, but his results contradict the basic beliefs of the classical economists. The situation is even worse with Samuelson. He says "On Mill's behalf" and then goes on to demonstrate that Mill's understanding of the future was wrong. I think that what happened was that these authors were more interested in drawing the logical consequences of the classical assumptions rather than engaging in a detailed analysis of whether the classical economists correctly understood all the implications of their assumptions.

In respect to the second question, it is certainly true that for general models of this type the rate of profit will fall to its minimum level. What happened in the two cited cases is that the authors chose accidentally, and without justification, the two special cases where this does not happen: Johansen has a unitary elasticity of substitution while Samuelson has no capital augmenting technical progress.

With respect to the classical literature the contribution of this paper is two fold: first it shows that the conclusions of the modern writers are

not correct generally, and second, it shows that if one takes the model with neoclassical capital as the basis for classical thinking, then these authors, although they were unaware of the necessary reasoning, had accidentally reached the generally correct conclusion.

I think that the result has implications beyond the characteristics of the classical model. It would seem that most economists, if asked why the rate of profit has not consistently fallen, would point to the rapid technical progress in agriculture. Eltis (p.280) states

...technical progress has raised productivity enormously in both industry and agriculture and there has been no tendency for a rising relative cost of food to squeeze profits in the manner that Ricardo and Mill expected.

It may be true that the rate of profit has not fallen consistently, but the result of this paper implies that this can not be attributed to rapid progress in agriculture and is, thus, rather mysterious.

With regard to the Marxist literature, in the first place Marx's central idea is well-known: He thought that capitalism would fall and be replaced by socialism. He further held that capitalist development would be characterized by the following "historical tendencies": A rising capital-labour ratio, a rising share of capital and a falling rate of profit. In addition he seems to have initially thought that the wage would stay at subsistence but later changed his mind about this. These tendencies play important roles in Marx's various theories (never well worked out) of the end of capitalism<sup>4</sup>. However the focus of interest has been on Marx's theory of the falling rate of profit.

Marx thought that, in a temporary fashion, a shortage of labour could cause the wage to rise and the rate of profit to fall, but that the long run fall in the rate of profit would be due to firms choosing progressively more capital intensive means of production<sup>5</sup>. This dichotomy has given rise to two distinct "lines of thought". The first, which is loosely connected with the idea of the profit squeeze, attempts to explain the

<sup>4</sup> The phrase "historical tendencies" was coined by Duménil and Lévy (2003). Their list is slightly longer than the one given above. They emphasize that a constant share of capital is also consistent with Marx's writing. I have abbreviated the list and chosen the rising share of capital because these are the tendencies that carry the weight in Marx's theories of the end of capitalism. See Petith (2002) for a summary of these theories.

<sup>5</sup> Marx explains that the fall in the rate of profit is due to the technical choices of the firms and "Nothing is more absurd, for this reason, than to explain the fall in the rate of profit by a rise in the rate of wages, although this may be the case by way of an exception." (1984, p.240). A bit further along (on p.256) he explains this exception: "...the competitive struggle (among capitalists) is accompanied by a temporary rise in wages and a resultant further temporary fall in the rate of profit."

fall in the rate of profit in industrialized countries that started in the late sixties and continued into the middle eighties mainly in terms of rising wages<sup>6</sup>. The second, which I will refer to as the technical choice school, attempts to give a general explanation of a long run fall in the rate of profit when this is the result of firms' technical decisions, not pressure from the labour market. Although a natural background assumption would seem to be that of a constant wage, this has been discarded for two reasons: first, one can find some justification for a rising wage in Marx's writings, and second, Marx's own argument about firms' technical choices has been shown by Okishio (1961) to imply a rising wage. Thus the objective of the technical choice school seems to have become to explain a long run fall in the rate of profit in terms of the technical choices of firms where lack of labour market pressure is evidenced by a non-rising labour share<sup>7</sup>.

There are three distinct contributions to this school. First, Skott (1992) and Michl (1994) use a monopolistic competition setting and a Kalecki type wage determination to show that the rate of profit will fall as firms adjust slowly to an optimal capital-labour ratio. Here, since the models have a steady state, the fall in the rate of profit comes to an end. Second, Skillman (1997) has a matching and bargaining model of the labour market where the individual outcomes depend on economy wide determined outside options. In the presence of capital using labour saving innovations, firms make technical choices that are profit maximizing at the old outside options but change these in a way that the labour share remains constant and the rate of profit falls. Here there is no natural end to the fall in the rate of profit. Finally Duménil and Lévy, in a sequence of papers culminating in (2003) have a model with a steady state in which the share of labour is constant and the rate of profit falls. The model has an endogenous labour supply, a link between the rates of growth of employment and the wage, and an induced innovation mechanism in which the factor shares determine the rates of change of the input coefficients. Their contributions are notable because they span the two lines of thought: In *The Economics of the Profit Rate* (1993, chapter 15) they explain the post 1965 fall in the rate of profit in terms of an early version of the model and, in the latest version, primarily thanks to the induced innovation mechanism, they manage to generate all of the historical tendencies.

The present paper belongs to the technical choice school since the assumption of an infinitely elastic labour supply means that labour

<sup>6</sup> These writings are surveyed in Howard and King (1992, chapter 16). A recent contribution which contains a critical survey is Brenner (1998).

<sup>7</sup> See Duménil and Lévy (2003, p.206) for a detailed justification of this in terms of Marx's writings.



market pressure is absent. In this area, its contribution is to show that by adding land, the historical tendencies can be generated without having a rising wage. This is important both for understanding what causes a falling rate of profit and for the coherence of Marx's overall view. First one can take the labour market assumption as an extreme case where there is no pressure on profits from this quarter. This means that when we observe a falling rate of profit (as in the period that started in the late sixties), it is perfectly possible that it may have little to do with a rising wage. Second, the historical tendencies are important, not for themselves but because of the roles they played in Marx's various accounts of the end of capitalism. Many of these stories involve the miserable conditions of the workers. For example, the falling rate of profit causes the capitalists to increase pressure on the workers, which, in turn, occasions a rise in social conflict. But, as this example makes clear, the force of these accounts is much weakened if they are set in the context of rising wages. Thus the importance here is that the present paper shows how the historical tendencies can be generated without, at the same time, calling into doubt their *raison d'être*.

The Duménil-Lévy induced innovation mechanism has a paradoxical relation to the present paper that can be illustrated by looking at Foley (2003). Foley has a model that is very similar to the one of this paper with output produced by land and a capital-labour aggregate and an elasticity of substitution less than one. Yet the model converges to a steady state with a constant positive rate of profit. This seems to contradict the result of the present paper; what happened? Foley incorporates a version of the Duménil-Lévy induced innovation mechanism in his model. This, in turn, implies that the rate of capital augmenting technical progress approaches zero<sup>8</sup> so that one of the two assumptions of the present model is violated. The paradox is that if one wants to generate the historical tendencies as is done in the present model, one has to deny the validity of just the induced innovation mechanism that was responsible for them in the Duménil-Lévy model.

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<sup>8</sup> This is a simplification: there are two distinct mechanisms, and Foley has two versions. The first mechanism is associated with Kennedy (1964), involves the rates of factor augmenting technical progress, and is set in the context of a Solow growth model. Drandakis and Phelps (1966) showed that this implies that the rate of capital augmenting technical progress approaches zero. The Duménil-Lévy mechanism involves rates of change of the input coefficients and is set in the context of a classical fixed coefficient model. In one of Duménil and Lévy's cases the change in the capital input coefficient approaches zero. Foley has a classical and a neoclassical version. In both of these, if land is not considered a free good but is priced at its marginal product, then the rate of capital augmenting technical progress or the rate of change of the capital input coefficient approaches zero and the rate of profit approaches a positive constant.

The paper is organized as follows. Section 2 presents the model. This generates a single non-autonomous differential equation in the aggregate-land ratio. The asymptotic form of this equation is solved and the main result is deduced from this solution. Section 3 provides an intuitive explanation of the main result. Finally Section 4 shows that the path generated by the asymptotic form approaches that generated by the original differential equation.

## 2. Model and Result

There are two factors of production, each measured in effective units: Land,  $M \equiv \underline{M}e^{\delta t}$  where the quantity of physical units  $\underline{M}$  is set equal to 1,  $\delta \geq 0$  is the rate of land augmenting technical progress and  $t$  is time; and a Cobb-Douglas capital-labour aggregate  $X \equiv K^\beta L^{1-\beta} e^{\gamma t}$ , where  $K$  and  $L$  are capital and labor in physical units,  $\beta$  is a constant  $0 < \beta < 1$  and  $\gamma \geq 0$  is the rate of aggregate augmenting technical progress<sup>9</sup>.  $x \equiv X/M$  is the aggregate-land ratio in terms of effective units:

$$x = K^\beta L^{1-\beta} e^{(\gamma-\delta)t}. \quad (1)$$

There is a single good, output  $Y$  is produced by a CES production function in  $M$  and  $X$ ,  $Y = [\alpha X^{-\rho} + (1-\alpha) M^{-\rho}]^{-\frac{1}{\rho}}$  where  $0 < \alpha < 1$  is a constant and the elasticity of substitution between  $M$  and  $X$  is  $\sigma = \frac{1}{1+\rho}$ ,  $-1 \leq \rho \leq \infty$ . Bringing  $M$  outside the brackets gives

$$Y = f(x) \underline{M} e^{\delta t}, \quad f(x) \equiv \frac{c_1 x}{(c_2 + x^\rho)^{\frac{1}{\rho}}}, \quad \underline{M} = 1, \quad (2)$$

where  $c_1 = 1/(1-\alpha)^{\frac{1}{\rho}}$  and  $c_2 = \alpha(c_1)^\rho$ .  $f(x)$  is the ratio of output to land in effective units which depends on  $x$ .

The supply of labour is infinitely elastic at the subsistence wage  $w$ . The demand, and thus the quantity of labour, is determined so that the marginal product of labour is equal to the wage. The marginal product of the aggregate is  $f'(x)$  ( $f' \equiv df/dx$ ) and  $x e^{\delta t}$  is its quantity, so  $x f'(x) e^{\delta t}$  is the payment it receives. Because the aggregate is Cobb-Douglas, labour receives  $(1-\beta)$  of this payment. Thus

$$L = \frac{1-\beta}{w} x f'(x) e^{\delta t} = \frac{1-\beta}{w} \frac{c_1 c_2 x}{(c_2 + x^\rho)^{1+\frac{1}{\rho}}} e^{\delta t}. \quad (3)$$

<sup>9</sup> Since the aggregate is Cobb-Douglas, it can be thought of as capital augmenting.

From (1) and (3)

$$K = x^{\frac{1}{\beta}} L^{-\frac{1-\beta}{\beta}} e^{\frac{\delta-\gamma}{\beta}t} = \left(\frac{1-\beta}{\beta} c_1 c_2\right)^{-\frac{1-\beta}{\beta}} \frac{x}{(c_2 + x^\rho)^{\frac{1}{\rho}(1-\mu)}} e^{(\delta-\phi)t} \quad (4)$$

where  $\mu = \frac{1+\rho(1-\beta)}{\beta} > 1$  if  $\rho > 0$  and  $\phi = \frac{\gamma}{\beta} > 0$  if  $\gamma > 0$ . Thus  $L$  and  $K$  are given as functions of  $x$  and  $t$ .

Capitalists own both the stock of capital and the land. They receive the output, pay the wage to the workers and get capital gains on the land,  $\dot{P}M$ , where  $P$  is the price of land in physical units and  $\dot{P} \equiv dP/dt$ . It is assumed that they save their entire income<sup>10</sup>. Thus savings are  $Y - wL + \dot{P}M$ . The change in wealth is  $\dot{K} + \dot{P}M$ . Setting the two equal gives

$$\dot{K} = Y - wL. \quad (5)$$

Thus the assumption that capitalists save all eliminates capital gains and allows the model to be solved without taking the path of the price of land into account.<sup>11</sup>

The model reduces to a non-autonomous differential equation in  $x$ . From (1)

$$\hat{x} = \beta \hat{K} + (1-\beta)\hat{L} + \gamma - \delta \quad (6)$$

where  $\hat{x} \equiv \dot{x}/x$  and so on. From (3)

$$\hat{L} = \delta + \frac{c_2 - \rho x^\rho}{c_2 + x^\rho} \hat{x}. \quad (7)$$

From (2), (3) and (5),  $\dot{K}$  is a function of  $x$  and  $t$ . Dividing this expression by (4) to get  $\hat{K}$  gives

$$\hat{K} = c_1^{\frac{1}{\beta}} c_2^{\frac{1-\beta}{\beta}} \left(\frac{1-\beta}{w}\right)^{\frac{1-\beta}{\beta}} \frac{x^\rho + \beta c_2}{(x^\rho + c_2)^{\frac{1}{\beta}(1+\frac{1}{\rho})}} e^{\phi t}. \quad (8)$$

<sup>10</sup> One might object that the saving behavior of capitalists should be determined by intertemporal optimization. There are two questions: what is the effect of continued saving on the rate of profit, and what is the effect of the temporal pattern of the profit rate on saving? This paper seeks to answer the first question only.

<sup>11</sup> Once one has solved the model for the path of  $x$ , one can use the condition that the rates of return on land and physical capital must be equal (equation 16 below) to solve for the path of the price of land. This has the peculiarity that the price of land becomes negative in finite time. The model can be made consistent by assuming that the world ends in finite time and then choosing the initial price of land so that land has zero price as the world ends. The theorem of this paper is then stated in terms of having the time the world ends approach infinity. This is set out in detail in Petith (2005). It is beyond the scope of the present paper to explore this issue in depth.

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Substituting (7) and (8) into (6) gives

$$\dot{x} = F(x)e^{\phi t} + G(x) \text{ with } x(t_0) = X_o > 0 \quad (9)$$

where

$$F(x) \equiv \frac{a\mu(x^\rho + \beta c_2)}{(c_2 + x^\rho)^{\frac{\mu}{\rho}}(c_2 + \mu x^\rho)} x,$$

$$G(x) \equiv (\phi - \delta) \frac{c_2 + x^\rho}{c_2 + \mu x^\rho} x,$$

$$a \equiv \frac{(c_1 c_2^{1-\beta})^{\frac{1}{\beta}}}{\mu} \left( \frac{1-\beta}{w} \right)^{\frac{1-\beta}{\beta}},$$

and it is supposed that the initial value of  $x$  is positive.

Let  $x(t)$  be the continuous solution to (9)<sup>12</sup>. Lemma 1 gives some of its characteristics,

LEMMA 1. *If  $x$  is a continuous solution to (9) and  $x(t_0) > 0$ , then i)  $x(t) > 0$  for  $t_0 \leq t < \infty$  and if in addition  $\gamma > 0$ , then ii)  $x(t)$  is unbounded for  $t_0 \leq t \leq \infty$ .*

*Proof.* First note that  $x(t) > 0$  for  $t_0 \leq t < \infty$ . Suppose  $x(t) = 0$  for some finite value of  $t$ . Let  $t_*$  be the first such value. Then  $x(t) \geq 0$ ,  $t_0 \leq t \leq t_*$ . This means that  $F(x)/x$  and  $G(x)/x$  are bounded below on this domain and there is an  $A$  such that

$$\frac{F(x(t))}{x(t)} e^{\phi t} + \frac{G(x(t))}{x(t)} > A, \quad t_0 \leq t \leq t_*$$

so that  $\dot{x}(t) > Ax(t)$  on this domain. Integrating from  $t_0$  to  $t_*$  and using both sides of the inequality as exponents gives

$$x(t_*) > x(t_0) e^{A(t_*-t_0)} > 0,$$

contradiction.

Suppose  $x(t) < x^*$  for  $t_0 \leq t \leq \infty$ . From the continuity of  $x(t)$ ,  $x(t) \geq 0$ ,  $t_0 \leq t \leq \infty$ . Thus there is an  $\underline{F} > 0$  and  $\underline{G}$  such that  $F(x(t))/x(t) > \underline{F}$  and  $G(x(t))/x(t) > \underline{G}$  for  $t_0 \leq t \leq \infty$ . Since  $\gamma > 0$  implies  $\phi > 0$ , there exists  $t'$  and a  $B > 0$  such that

$$\dot{x}(t) = F(x(t))e^{\phi t} + G(x(t)) > [\underline{F}e^{\phi t} + \underline{G}] x(t) > Bx(t)$$

for  $t > t'$ . Integrating as above,

$$x(t) > x(t') e^{B(t-t')}$$

which contradicts the supposed bound on  $x(t)$ . □

<sup>12</sup> See Petith (2002) for a proof of the existence and uniqueness of  $x(t)$  on  $[t_0, \infty)$ .

It is instructive to consider what happens to (9) as  $x \rightarrow \infty$  when  $\rho > 0$ . Define  $\tilde{F}(x)$  and  $\tilde{G}(x)$  by

$$F(x) = \frac{a}{x^{\mu-1}} \tilde{F}(x) \text{ and } G(x) = \frac{\phi - \delta}{\mu} x \tilde{G}(x). \quad (10)$$

When  $\rho > 0$  these functions satisfy

$$\lim_{x \rightarrow \infty} \tilde{F}(x) = \lim_{x \rightarrow \infty} \tilde{G}(x) = 1. \quad (11)$$

Consider the equation that arises when the asymptotic values of these functions are substituted into (9):<sup>13</sup>

$$\dot{x} = \frac{ae^{\phi t}}{x^{\mu-1}} + \frac{\phi - \delta}{\mu} x, \text{ with } x(T) = X. \quad (12)$$

In order to determine the solution to (12), substitute the function  $y(t)$  for  $x(t)$  with  $y(t) = a\mu \frac{e^{\phi t}}{(x(t))^\mu}$ . The function  $y(t)$  satisfies the following differential equation,  $\dot{y} = y(\delta - y)$ , which can be easily integrated,  $\bar{y}(t) = \frac{\delta}{1 - Ce^{-\delta t}}$ . Thus, the solutions of (12) can be determined.

$$\bar{x}(t; a, \delta, T, X) = \left[ \frac{a\mu}{\delta} e^{\phi t} (1 - C(a, \delta, T, X) e^{-\delta t}) \right]^{\frac{1}{\mu}} \quad (13)$$

in which the constant  $C(a, \delta, T, X)$  is determined by the initial condition  $X = \bar{x}(T; a, \delta, T, X)$ . Also the function  $\bar{\bar{x}}(t)$  can be defined whose asymptotic behavior is identical to that of  $\bar{x}(t)$  when  $t$  tends to infinity:

$$\bar{\bar{x}}(t) \equiv \left( \frac{a\mu}{\delta} e^{\phi t} \right)^{\frac{1}{\mu}}. \quad (14)$$

It would seem likely that the solution to (9),  $x(t)$ , would tend to  $\bar{\bar{x}}(t)$  as  $t \rightarrow \infty$ . Indeed this is the case as will be shown below. This is stated as Lemma 2.

LEMMA 2. *Let  $\rho > 0$  and  $\gamma > 0$ , then*

$$\lim_{t \rightarrow \infty} |x(t) - \bar{\bar{x}}(t)| = 0.$$

The elements of the historical tendencies may now be defined.  $k \equiv K/L$  is the capital-labour ratio and  $s \equiv 1 - \frac{wL}{Y}$  is the share of income

<sup>13</sup> It is convenient to have two meanings for  $X$ . Previously it was the capital-labour aggregate. Here it is the value that  $x(t)$  assumes when  $t = T$ . The meaning of  $X$  will be indicated each time it is used.

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received by the capitalists. Next consider the rate of profit. Since capitalists' share of the income of the aggregate is  $\beta$ , as in the justification of (3),

$$rK = \beta x f'(x) e^{\delta t} \quad (15)$$

where  $r$  is the marginal product of capital. The rate of profit  $R$  is defined as total capitalist income divided by the value of factors of production owned by the capitalists:

$$R \equiv \frac{Y - wL + \dot{P}M}{K + PM}.$$

Since capitalists hold both capital and land, the rate of return on both of these assets, when calculated in terms of the good, must be the same:

$$r = \frac{\frac{\partial Y}{\partial M} + \dot{P}}{P}. \quad (16)$$

It is easily shown that (16) implies that  $R = r$ . Thus from this point on  $r$  will be taken as the rate of profit. From (15)  $r = \beta x f'(x) e^{\delta t} / K$ .

The result of the paper can now be proved. It states that as  $t$  approaches infinity, the capital-labour ratio approaches infinity, the capitalist share approaches one, and the rate of profit approaches zero.

**THEOREM 3.** *Let  $\rho > 0$  and  $\gamma > 0$ . Then  $k(t) \rightarrow \infty$ ,  $s(t) \rightarrow 1$ , and  $r(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof.* Let  $L(t)$  and  $K(t)$  be the values of  $L$  and  $K$  derived from (3), (4) and the function  $x(t)$ . From (3) and (15)

$$r(t) = \frac{w\beta}{1 - \beta} \frac{1}{k(t)}. \quad (17)$$

From (2) and (3)

$$s(t) = \frac{\beta c_2 (x(t))^{-\rho} + 1}{c_2 (x(t))^{-\rho} + 1}.$$

From (3) and (4)

$$k(t) = c_3 [c_2 + (x(t))^\rho]^{\frac{\mu+\rho}{\rho}} e^{-\phi t}$$

where  $c_3 = \frac{w}{\beta} (\frac{1-\beta}{\beta} c_1 c_2)^{-\frac{1}{\beta}}$ . Thus, since  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$  from Lemma 2,

$$k(t) \rightarrow c_3 x(t)^{\mu+\rho} e^{-\phi t} \text{ as } t \rightarrow \infty.$$

Let  $d(t)$  be the distance between  $x(t)$  and  $\bar{x}(t)$  that is

$$d(t) \equiv x(t) - \left(\frac{a\mu}{\delta} e^{\phi t}\right)^{1/\mu}$$

where  $d(t) \rightarrow 0$  as  $t \rightarrow \infty$  from Lemma 2. Then

$$\begin{aligned} k(t) &\rightarrow c_3 \left[ \left( \frac{a\mu}{\delta} \right)^{1/\mu} e^{\frac{\phi}{\mu}t} + d(t) \right]^{\mu+\rho} e^{-\phi t} \\ &= c_3 \left[ \left( \frac{a\mu}{\delta} \right)^{1/\mu} e^{\frac{\rho}{\mu(\mu+\rho)}\phi t} + d(t) e^{-\frac{1}{\mu+\rho}\phi t} \right]^{\mu+\rho} \\ &\rightarrow c_3 \left( \frac{a\mu}{\delta} \right)^{\frac{\mu+\rho}{\mu}} e^{\frac{\rho}{\mu}\phi t} \text{ as } t \rightarrow \infty. \end{aligned} \quad (18)$$

Thus since  $\rho > 0$  and since  $\gamma > 0$  implies  $\phi > 0$ ,  $k(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Thus  $r(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Finally, since by Lemma 2  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ,  $\rho > 0$  means that  $s(t) \rightarrow 1$  as  $t \rightarrow \infty$ .  $\square$

### 3. A Heuristic Description of the Result<sup>14</sup>

In this description land and the aggregate are always expressed in effective units while labour and capital are in physical units. The first step is to understand why the capital-labour aggregate in effective units,  $X$ , grows faster than land,  $M$ , in the same units. It must do this in order to keep the marginal product of labour constant because the aggregate admits technical progress. This can be seen as follows: suppose the rate of growth of the aggregate was less than or equal to that of land,  $\hat{X} \equiv \delta' \leq \delta$ .

a.  $\hat{Y} \geq \delta'$  by constant returns to scale and, since there is a constant savings ratio,  $\hat{Y} = \hat{K}$  asymptotically so that  $\hat{Y} = \hat{K} \geq \delta'$  asymptotically.

b. Since  $\hat{X} = \beta\hat{K} + (1-\beta)\hat{L} + \gamma$ ,  $\hat{L} \leq \delta' - \frac{\gamma}{1-\beta}$ . That is, since capital is growing at least as fast as the aggregate, labour must grow more slowly to compensate for the technical progress.

c. But in this case the marginal product of labour rises both because the capital to labour ratio increases and the ratio of the aggregate to land falls. Since this is impossible, the aggregate must grow faster than land.

To put this in a nutshell: If the aggregate does not grow faster than land, then capital will grow at least as fast as the aggregate, labour will grow slower to compensate for the technical progress of the aggregate and the impossible wage growth will occur.

<sup>14</sup> The arguments of this section are only approximate. It is frequently stated that the rates of growth of the variables approach limits. This is convenient for an intuitive discussion and may well be the case, but it has not been formally demonstrated. The problem is that Lemma 2 does not imply that  $\hat{x}(t) \rightarrow \hat{\bar{x}}(t)$  as  $t \rightarrow \infty$ .

The second step is to understand why this implies that capital must grow faster than labour: Since  $\sigma < 1$ , asymptotically, the slowest growing factor dominates so that  $\hat{Y} = \delta$ . Again asymptotically, because of the constant savings ratio,  $\hat{K} = \hat{Y}$ . Finally from the condition that the wage is constant,  $\hat{L} = \delta - \rho\hat{x}$ , which can be understood as follows: If  $x$  were constant, then the payment to the aggregate would grow at  $\delta$  and, since labour receives a constant proportion of this, the labour supply would have to grow at  $\delta$  to keep the wage fixed. Since  $\sigma < 1$  the growth of  $x$  will reduce the payment to the aggregate at an asymptotic rate of  $\rho\hat{x}$  so that the payment to labour grows at  $\delta - \rho\hat{x}$  and the labour supply must grow at this rate to keep the wage constant. Thus it is a combination of the relative growth of the aggregate,  $\sigma < 1$ , and the constant wage that forces capital to grow faster than labour.<sup>15</sup>

Finally, what forces drive the historical tendencies? First, the aggregate grows faster than land because of technical progress in the aggregate. Because of this, with  $\sigma < 1$ , in order to keep the wage constant labour must grow more slowly than capital. This establishes that  $k \rightarrow \infty$ . Second, since both the share of capital in the aggregate and the reward to labour are constant, the faster growth of capital must be compensated for by a fall in its reward. This establishes that  $r \rightarrow 0$ . (Alternatively, the rise in  $K/L$  with the marginal product of labour fixed forces the marginal product of capital to fall.) Finally, since  $\sigma < 1$ , the faster growth of the aggregate forces its share and that of labour to zero. This establishes that  $s \rightarrow 1$ .

With this detailed heuristic account, the reader may be in danger of not seeing the forest for the trees. A simpler, less accurate explanation is the following. Think in terms of a model with only capital and land. Generally, when one adds factor augmenting technical progress to a factor that can be accumulated, like capital, the result is explosive growth in the sense that the rate of growth increases over time. But in our simplified model, with elasticity less than one, this is counteracted by the slower growth of land in efficiency units. The result is a balance between these two forces in which the growth of capital, in efficiency units, is not explosive but is more rapid than that of land with the consequence of a continually falling rate of profit.

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<sup>15</sup> With slightly more effort one can see why  $\hat{x} = \gamma/[1 + \rho(1 - \beta)]$ : Put  $\hat{K} = \delta$  and then the above expression for  $\hat{L}$  into (7).



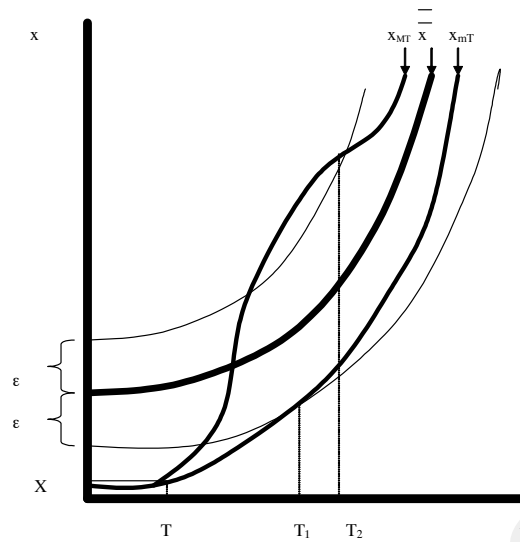


Figure 1. Illustration of lemmas 5 and 6

#### 4. Proof of Lemma 2

This section provides the proof of Lemma 2, but only for the case of  $\delta - \phi > 0$ . This will be called Lemma 6. This is the most difficult and also the most interesting case since it is the one in which the rate of technical progress in land  $\delta$  can be unboundedly large. The proof of the other case,  $\delta - \phi \leq 0$ , is set out in Petith (2002). Lemma 6 states that  $x(t)$ , the solution to (9), approaches  $\bar{x}(t)$ , given by (14), as  $t \rightarrow \infty$ . The proof proceeds in two steps and can be read from figure 1. Lemma 5 uses Chaplygin's theorem to establish that  $x$  lies between two bounding functions  $x_{mt}$  and  $x_{MT}$ . Then Lemma 6 shows that these two bounding functions eventually enter an  $\epsilon$  tube that surrounds  $\bar{x}$ ; thus  $x$  is asymptotically equivalent to  $\bar{x}$ .

First Chaplygin's theorem is stated:

**THEOREM 4.** <sup>16</sup> Let  $x(t)$  be the solution to the equation  $\dot{x} = g(x, t)$ ,  $x(T) = X$ , and let  $x_{mt}(t)$  and  $x_{MT}(t)$  be two bounding functions with  $x_{mT}(T) = x_{MT}(T) = X$ . If the differential inequalities

$$\dot{x}_{mT}(t) - g(x_{mT}(t), t) < 0$$

<sup>16</sup> See Mikhlin and Smolitskiy (1967, pp. 9-12) or Zwillinger (1989, pp. 388-391).

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$$\dot{x}_{MT}(t) - g(x_{MT}(t), t) > 0$$

hold for  $t > T$ , then

$$x_{mT}(t) < x(t) < x_{MT}(t)$$

for all  $t > T$ .

Next the bounding functions are constructed by taking the solutions to modified forms of (12). First modify (12) as

$$\dot{x} = \frac{a}{x^{\mu-1}} e^{\phi t} + m \frac{\phi - \delta}{\mu} x = \frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta_m}{\mu} x, \quad x(T) = X \quad (19)$$

where  $\delta_m = \delta m + (1 - m)\phi$ . Then take  $\bar{x}_{mT}(t)$  as the solution to this equation with  $\delta$  in (13) replaced by  $\delta_m$ . Next modify (12) as

$$\dot{x} = M \left( \frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta}{\mu} x \right) = \frac{a_M}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta_M}{\mu} x, \quad x(T) = X \quad (20)$$

where  $a_M = Ma$  and  $\delta_M = \delta M + (1 - M)\phi$ . Then take  $\bar{x}_{MT}(t)$  as the solution to this equation with  $a$  and  $\delta$  in (13) replaced by  $a_M$  and  $\delta_M$ .

Now Lemma 5 may be proved.

LEMMA 5. *Let  $x(t)$  be the solution to (9), let  $\rho > 0$ , and  $\gamma > 0$  and take  $\phi - \delta < 0$ . For any  $\tilde{x}$  there exists a  $T$  such that  $x(T) > \tilde{x}$  and*

$$\bar{x}_{mT}(t) < x(t) < \bar{x}_{MT}(t), \quad t > T$$

where

$$\bar{x}_{mT}(t) = \bar{x}(t; a, \delta_m, X(T), T), \quad m = \tilde{G}(x(T))/\tilde{F}(x(T))$$

and

$$x_{MT}(t) = \bar{x}(t; a_M, \delta_M, x(T), T), \quad M = \tilde{G}(x(T)).$$

*Proof.* The properties of the bounding functions depend on those of  $\tilde{F}(x)$  and  $\tilde{G}(x)$ . It is clear that

$$\tilde{G}(x)/\tilde{F}(x) > 1, \quad (21)$$

and that there exists an  $\bar{x}$  such that

$$\tilde{F}'(x) < 0, \tilde{G}'(x) < 0 \text{ and } (\tilde{G}(x)/\tilde{F}(x))' < 0 \text{ for } x > \bar{x}. \quad (22)$$

Next a condition on the derivatives of  $\bar{x}_{mT}$  and  $\bar{x}_{MT}$  is given. For the given  $\tilde{x}$  choose  $T$  so that  $x(T) > \tilde{x}$ , so that  $x(T) > \bar{x}$  of (22) and

so that  $\dot{x}(T) > 0$ . This is possible since, by Lemma 1  $x(t)$  is unbounded above. Since  $\dot{x}(T) > 0$ , writing (9) in terms of  $\tilde{F}(x)$  and  $\tilde{G}(x)$  gives

$$\frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi - \delta}{\mu}x\tilde{G}(x) = \dot{x}(T) > 0, \quad x = x(T), \quad (23)$$

by (23)

$$\dot{\bar{x}}_{mT}(T) = \frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi - \delta}{\mu}x\frac{\tilde{G}(x)}{\tilde{F}(x)} > 0, \quad x = x_{mT}(T)$$

and finally

$$\dot{\bar{x}}_{mT}(t) > 0, \quad t \geq T. \quad (24)$$

The last inequality follows by differentiating (13) with respect to time to get

$$\dot{\bar{x}}(t) = \frac{1}{\mu}\bar{x}^{\frac{1}{\mu}-1}\frac{a\mu}{\delta}e^{\phi t}\left[\phi + (\delta - \phi)C(\cdot)e^{-\delta t}\right]. \quad (25)$$

Replacing  $\delta$  with  $\delta_m$  shows that  $\dot{\bar{x}}_{mT}$  is either positive (if  $(\delta - \phi)C(\cdot) > 0$ ) or increasing in  $t$ .

Also

$$\begin{aligned} \dot{\bar{x}}_{mT}(T) &= \tilde{G}(x)\left(\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi - \delta}{\mu}x\right) \\ &> \tilde{G}(x)\left(\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi - \delta}{\mu}x\right) > 0, \quad \text{for } x = x_{mT}(T) \end{aligned}$$

where the first inequality follows from (21) and the second from (23), and, as above, from (25)

$$\dot{\bar{x}}_{mT}(t) > 0, \quad t \geq T. \quad (26)$$

Finally it is shown that the conditions of Chaplygin's theorem are satisfied.

$$x_{mT}(T) = x(T) = x_{mT}(T)$$

by construction.

$$\bar{x}_{mT}(t) > \bar{x}_{mT}(T), \quad \bar{x}_{mT}(t) > \bar{x}_{mT}(T), \quad t > T \quad (27)$$

by (24) and (26).

$$\begin{aligned} \dot{\bar{x}}_{mT}(t) &= \frac{a}{x^{\mu-1}}e^{\phi t} + m\frac{\phi - \delta}{\mu}x < \left(\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\tilde{G}(x)}{\tilde{F}(x)}\frac{\phi - \delta}{\mu}x\right)\tilde{F}(x) \\ &= \frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi - \delta}{\mu}x\tilde{G}(x), \quad \text{for } x = \bar{x}_{mT}(t), \quad t > T \end{aligned}$$

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by (11), (21), (22), and (27).

$$\begin{aligned}\dot{\bar{x}}_{MT}(t) &= M\left(\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi - \delta}{\mu}x\right) > \left(\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi - \delta}{\mu}x\right)\tilde{G}(x) \\ &= \frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi - \delta}{\mu}x\tilde{G}(x) \text{ for } x = \bar{x}_{MT}(t), t > T\end{aligned}$$

by (21), (22) and (27). This completes the proof.  $\square$

Finally Lemma 6 is proved for the case of  $\phi - \delta < 0$ .

LEMMA 6. Let  $x(t)$  be the solution to (9), let  $\rho > 0$  and  $\gamma > 0$  and take  $\phi - \delta < 0$ . Then

$$\lim_{t \rightarrow \infty} |x(t) - \bar{x}(t)| = 0.$$

*Proof.* Choose  $\epsilon$  arbitrary but with  $\epsilon < 1$ ,  $\epsilon < \frac{4}{(\frac{\phi}{\delta} - 1)}$ . It must be shown that there is a  $T_\epsilon$  such that

$$(1 - \epsilon)\bar{x}(t) < x(t) < (1 + \epsilon)\bar{x}(t), \quad t > T_\epsilon.$$

Choose  $\tilde{x}$  large enough so that, for the  $T$  given by Lemma 5,

$$\frac{\tilde{G}(x)}{\tilde{F}(x)} < 1 + \frac{\epsilon/2}{1 - \frac{\phi}{\delta}}, \quad \tilde{G}(x) < 1 + \frac{1}{\frac{\phi}{\delta} \frac{1 + \epsilon/2}{\epsilon/4} - 1}, \quad \text{for } x \geq x(T). \quad (28)$$

Now apply Lemma 5. The proof is completed by showing that there exists a  $T_\epsilon$  such that

$$(1 - \epsilon)\bar{x}(t) < \bar{x}_{mT}(t), \quad \bar{x}_{mT}(t) < (1 + \epsilon)\bar{x}(t), \quad t > T_\epsilon.$$

Choose  $T_1 > T$  so that

$$(1 - \epsilon/2) < 1 - e^{-\delta_m t} C(a, \delta_m, x(T), T), \quad t > T_1.$$

(28), and the definitions of  $m$  and  $\delta_m$  imply

$$\begin{aligned}\frac{1}{1 + \epsilon/2} &< \frac{1}{m + (1 - m)\frac{\phi}{\delta}} = \frac{1}{\delta_m/\delta}. \\ (1 - \epsilon)^\mu &< (1 - \epsilon) < \frac{1}{1 + \epsilon/2}(1 - \epsilon/2) < \frac{1}{\delta_m/\delta}(1 - \epsilon/2), \\ (1 - \epsilon)^\mu \frac{a^\mu}{\delta} e^{\phi t} &< \frac{a^\mu}{\delta_m} e^{\phi t} (1 - e^{-\delta_m t} C(a, \delta_m, x(T), T)), \\ (1 - \epsilon)\bar{x}(t) &< \bar{x}_{mT}(t), \quad \text{for } t > T_1.\end{aligned}$$

Choose  $T_2 > T$  such that

$$1 - e^{-\delta_M t} C(a_M, \delta_M, x(T), T) < 1 + \epsilon/4, \text{ for } t > T_2.$$

(28) and the definition of  $M$  imply

$$\frac{M}{M + (1 - M)^{\frac{\phi}{\delta}}} < 1 + \epsilon/4.$$

This and the definitions of  $a_M$  and  $\delta_M$  give

$$\frac{a_M}{\delta_M} < \frac{a}{\delta}(1 + \epsilon/4).$$

$$\frac{a_M \mu}{\delta_M} e^{\phi t} (1 - e^{-\delta_M t} C(a_M, \delta_M, x(T), T)) < \frac{a \mu}{\delta} e^{\phi t} (1 + \epsilon/4)^2$$

$$< \frac{a \mu}{\delta} e^{\phi t} (1 + \epsilon) < \frac{a \mu}{\delta} e^{\phi t} (1 + \epsilon)^\mu, \quad t > T_2$$

$$\bar{x}_{MT}(t) < (1 + \epsilon) \bar{x}(t), \quad t > T_2.$$

The proof is completed by taking  $T_\epsilon = \text{Max}(T_1, T_2)$ .  $\square$

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