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Skewness as an Explanation of Gambling in Cumulative Prospect Theory.

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Abstract

Skewness of return has been suggested as a reason why agents might choose to gamble, ceteris paribus, in Cumulative Prospect Theory (CPT). We investigate the relationship between moments of return in two models where agents choices over uncertain outcomes are determined as in CPT. We illustrate via examples that in CPT theory, as with expected utility theory, propositions that agents have a preference for skewness may be invalid.

Keywords: Cumulative Prospect Theory; Exponential Value Function; Gambling
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Introduction:

A preference for skewness of return, ceteris paribus, is often suggested as a reason why agents might choose to gamble in the context of cumulative prospect theory, (CPT), (see Kahneman and Tversky (1979), Tversky and Kahneman (1992)). For example Barberis and Huang (2005) write, "through the probability weighting function, cumulative prospect theory investors exhibit a preference for (positive) skewness. Whilst Taleb (2004) suggests that "On the other hand, the value function of prospect theory documents a decreased sensitivity to both gains and losses, hence a marked overall preference for negative skewness".

A similar conjecture concerning skewness of return has also been made in the context of expected utility theory. For example Golec and Tamarkin (1998) assert "horse bettors accept low-return, high-variance bets because they enjoy the high skewness offered by these bets" and "we claim that bettors could be risk-averse and favour positive skewness, and primarily trade off negative expected return and variance for positive skewness.

In the context of expected utility theory the rationale for a positive skew preference is that the third derivative of the utility function of a globally risk-averse agent is positive, and approximation of the utility function to order three implies a valid trade-off between mean, variance and skewness of return, that is, preferring a higher third moment for two random variables having equal means and variances (Kraus and Litzenberger (1976)).

In fact the argument is in general incorrect for the expected utility maximiser. Brockett and Garven (1998) show that one can always construct two distributions with a given moment ordering for which neither stochastically dominates the other at any degree of stochastic dominance [i]

Given the conflicting statements concerning the preference of agents for skewness in cumulative prospect theory (and that the related conjectures in expected utility theory are incorrect), the purpose in this note is to investigate the relationship between moments of return, particularly skewness of return, in two models where agent's choices over uncertain outcomes are determined as in CPT.

The first model involves an agent undertaking a fixed stake gamble involving gains and losses. The second model is of a simple portfolio decision where the agent can purchase a safe asset paying an interest rate, h, or a bond that entitles you to participate in a lottery that pays X with probability p or zero otherwise. The key feature of the bond is that you do not loose your stake. You can cash in your holdings of the bond at any time. The UK government markets such a bond, called a National Premium Bond. It offers an expected rate of return that appears lower than the safe rate of return given by the interest rate paid on Treasury bills or Building Societies.[ii] Holdings of premium bonds are constrained by the UK government to a maximum of £30,000 and some 147,000 UK citizens hold this amount. Given the relatively large holdings by agents, with expenditure per year of some £2 billion, and a minimum purchase amount of £100, entertainment, as opposed to financial return, is, *apriori*, an unsatisfactory rationale for agents purchasing them. It is interesting to observe therefore that the UK government regards holdings of premium bonds as a form of saving but in the Budd report on Prevalence of Gambling in the UK (2001) they are considered as a component of gambling expenditure. Because the safe rate interest rate typically exceeds the expected rate of interest on bonds, we suggest that from the perspective of the standard expected utility model their holding constitutes an anomaly for many agents.

In the next section we set out our two models and analyse their implications. The final section of the note is a brief conclusion.

2.Some Analysis

We specify the value function as having the expo-power form which is a generalization of the power form employed by Tversky and Kahneman (1992)[iii]. . Defining reference point utility as zero, for a gamble to occur in CPT we require expected utility or value (EU) to be non-negative. For the expo-power value function expected value is given by

$$EU = w^{+}(p)(1-e^{-r\alpha s^{n}o^{n}}) - w^{-}(1-p)\lambda(1-e^{-\alpha s^{n}}) \geq 0 \qquad (1)$$

where the win-probability is given by p, and the functions $w^{+}(p)$ and $w^{-}(1-p)$ are non-linear s-shaped probability weighting functions, o are the odds and s the stake. r,α,n and λ are positive constants with $0<n\leq 1.$ The agent is risk-averse over gains and risk seeking over losses as assumed in CPT.

The value function in (1) has upper and lower bounds, as is required to resolve the St.Petersburg Paradox (see e.g. Bassett (1997)). The expo-power function also has the convenient property that as $\alpha \to 0,$ the value function approximates the power form employed by Tversky and Kahneman.

The degree of loss aversion, (LA) is defined by the ratio of the value of gain to the value of loss from a symmetric gamble, $o=1$ and is given by

$$LA = \frac{(1-e^{-r\alpha s^{n}})}{\lambda(1-e^{-\alpha s^{n}})} \qquad (2)$$

From (2), as stake size approaches zero loss aversion would require that $\frac{r}{\lambda}<1.$ and for large stake size $\frac{1}{\lambda}<1$. In order to ensure that the degree of loss aversion increases with stake size, we also require $r\geq 1.$

We assume that the probability weighting functions over gains and losses are as employed by Tversky and Kahneman (1992) and given by

$$w^+(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}} \quad , w^-(1-p) = \frac{(1-p)^\rho}{(p^\rho + (1-p)^\rho)^{\frac{1}{\rho}}} \qquad (3)$$

where δ and ρ are constants.

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The expected return from a one-unit stake gamble is defined by, μ^1, where

$$\mu^1 = po + (1-p)(-1) \text{ or } \mu = \mathrm{p}(1+\mathrm{o}) \qquad (4)$$

so that the return from an s stake gamble is

$$s\mu^1 = pos + (1-p)(-s) \qquad (5)$$

Skewness of return is given by

$$\mu_3 = \frac{s^3\mu^3(1-p)(1-2p)}{p^2} \qquad (6)$$

where $\mu = (1+\mu^1)$ and an actuarially fair bet is defined when $\mu^1 = 0$ or $\mu = 1$.

We assume that stake size is fixed at 10 units. We set the values of the parameters in the probability weighting functions at $\delta = 0.61$ and $\rho = 0.69$, those reported by Tversky and Kahneman (1992).

We let $n = 0.88,\ \mathrm{r} = 20,\ \mathrm{k} = 45\ \alpha = 0.00001.$

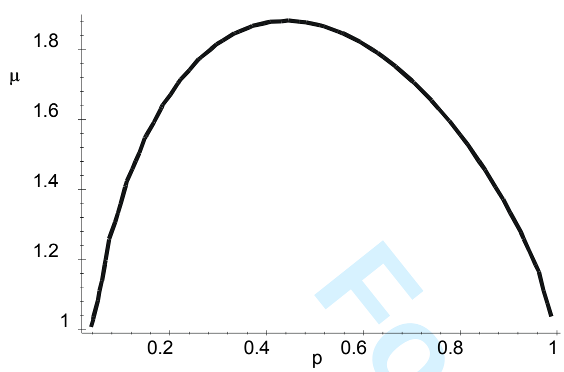
As noted above as $\alpha \to 0$, we obtain precisely the same specification of CPT and parameter values as reported by Tversky and Kahneman (1992). Because our choice of α is small our specification will exhibit the same features as that of Tversky and Kahneman for almost all of the probability range.

Substituting (4) and (3) into (1) we can write expected value solely as a function of expected return and win probability. Also using (6) we can write expected value solely as a function of skewness and expected return.

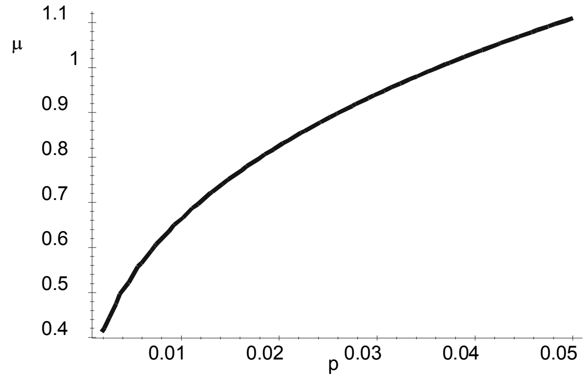
We plot the indifference curves between expected return and win probability and expected return and skewness of return for EU=0 in Figures 1(a) to 1(g).

Figures 1(a)-1(c) Indifference Curves between Expected Return and Win Probability

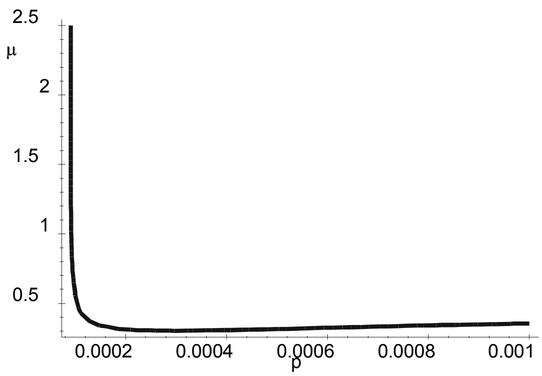
1(a) $0.05 < p \leq 1$



1(b) $0.001 < p < 0.05$

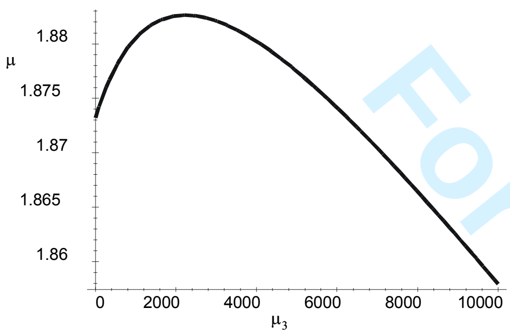


1(c) $0 \leq p < 0.001$

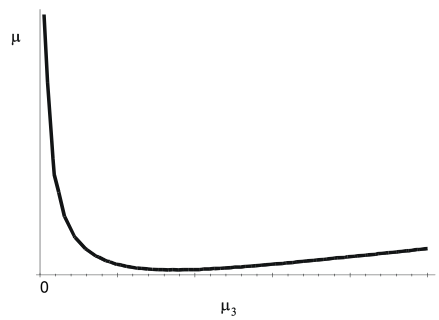


Figures 1(d)-1(g) Indifference Curves between Expected Return and Skewness of Return

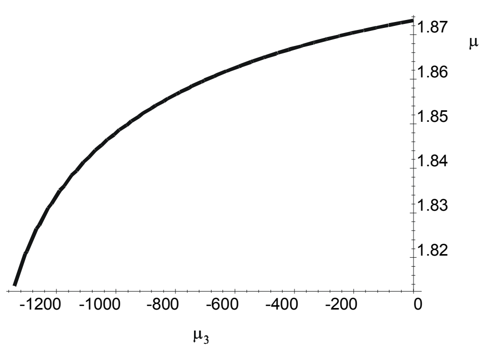
1(d) $0 \leq \mu_3 \leq 10000$



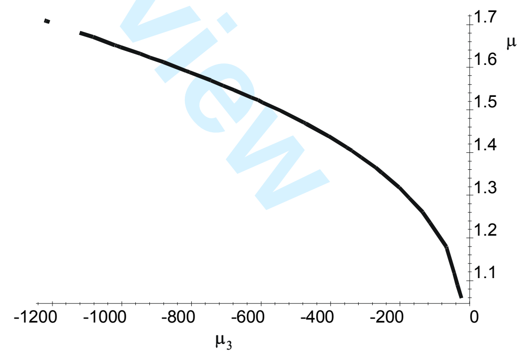
1(e) $10000 \leq \mu_3 \leq \infty$



1(f) $-1200 \leq \mu_3 \leq 0,\ 0.5 \leq p < \frac{2}{3}$



1(g) $-1200 \leq \mu_3 \leq 0,\ \frac{2}{3} \leq p \leq 1$



In Figures 1(a) and 1(b) we observe that the indifference curve exhibits a maximum in the range of actuarially fair bets and that the agent requires around $30 to stake. $10 at a probability of 0.5. These figures are the same as those reported by students in Kahneman and Tversky's (1979) experiments. We also note that due to probability distortion the agent will gamble at actuarially unfair odds on long shots.

In Figure1(c) we observe that at low enough probabilities the indifference curve exhibits a minimum and an asymptote. This is a resultant of the bounded ness of the value function. Ultimately boundedness implies that the agents will only take very long shot bets at actuarially fair odds and will ultimately reject some long shot bets regardless of the rate of return. This behaviour differs from that with a power function. It is discussed further in Cain et al (2005).

The indifference curve between expected return and skewness is plotted in Figures 1(d) to 1(g). The indifference curve has properties that reflect that for expected return and win probability. The important point to note is that the trade-off changes sign. For probabilities close to a half the trade-off is positive as shown in (1d) as the probability declines the trade-off exhibits a maximum and then a negative trade-off. For very low probabilities (very high positive skewness) the trade-off exhibits a minimum and the trade-off becomes positive as shown in Figure (1e). Clearly in some regions of positive skewness the agent requires a higher expected return to compensate for higher positive skewness. For probabilities between 0.5 and 2/3 the trade-off is negative, so the agent accepts more negative skewness and lower expected rates of return. For probabilities between 2/3 and 1 this is reversed, the agent exhibiting indifference between lower expected returns and less negative skewness. The Figures illustrate that there is no simple relationship between expected return and skewness of return for a fixed stake gamble in CPT.

In model 2 we assume that the agent has wealth of one unit to invest in a safe asset yielding h% and a premium bond yielding b%. The agent invests a and 1-a in the safe asset and premium bond respectively. In this model we assume that n=1 so as to allow a tractable analytic solution. The premium bond pays X with probability p so that $b = pX$.

Expected Value under CPT is given by:

$$EU = w^{+}(p)(1-e^{-\theta(ah+(1-a)X)}) + (1-w^{+}(p))(1-e^{-\theta ah}) \qquad (7)$$

The expected return of the portfolio is $ER = p(ah+(1-a)X)+(1-p)ah$

the variance, $\sigma^2 = \frac{(1-a)^2 b^2 (1-p)}{p}$ and skewness, $\sigma_3 = \frac{(1-a)^3 b^3 (1-2p)(1-p)}{p^2}$.

The optimal share of the portfolio in the safe asset, where $\frac{\partial EU}{\partial a} = 0, \frac{\partial^2 EU}{\partial a^2} < 0$, is given by

$$a = 1 - \frac{\ln \frac{w^{+}(p)\{X-h\}}{(1-w^{-}(p))h}}{\lambda X} \qquad (8)$$

We assume the parameters $\delta = 0.61, \theta = 0.3, h = 0.05, \theta = 0.3.$

Portfolio A has $X = 250, p = 0.0001$ and aq value of $a = 0.9613805$, which is optimal.

This gives the following moments for the portfolio;.

$\text{Expected Return(ER)} = 4.90345*(10^{-2})$

$\text{Variance } (\sigma^2) = 9.3207*(10^{-3})$ with $EU = 0.01767$.

$\text{skewness}(\sigma_3) = 8.9972*(10)^{-2}, \frac{\sigma_3}{(\sigma^2)^{1.5}} = 99.98$

For purposes of comparison the expected utility or value of the portfolio with a=1 and a=0 are $EU = 0.01489$ and $EU = 0.0036$ respectively.

Consider now a Portfolio B. This has $X = 1306.11, p = 8.742008*(10^{-6}), a = 0.975.$

These numbers generate the following moments for the portfolio .

$ER = 4.90345*(10^{-2})$

$\sigma^2 = 9.3207*(10^{-3}),$ with $EU = 0.0153266$

$\sigma_3 = 0.3043, \frac{\sigma_3}{(\sigma^2)^{1.5}} = 388.21$

Comparing portfolio's A and B we observe that they have identical means and variances but that B exhibits a higher positive skewness of return than portfolio A. Nevertheless the expected utility or value of portfolio A is greater than that of portfolio B. Consequently the example demonstrates that a KT agent can optimally prefer a portfolio with lower skewness than another when both exhibit the same expected return and variance.

Conclusions.

We have demonstrated with examples that in CPT, as with expected utility theory, it is not true to state that agents will prefer gambles that exhibit more positive skewness. In fact in model one we show that the agent can require a higher expected return to compensate for greater positive skewness. In the model two we show that a portfolio that has the same expected return and variance as another but lower positive skewness is preferred.

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Footnotes.

[i] They prove and show with simple examples that expected utility preferences never universally translate into moment preferences. In particular they show that there always exists two continuous unimodal random variables X and Y with identical means and variances but greater positive skew in the X variable. Nevertheless the preference relationship for the decision maker is exactly reversed, so that the expected utility of Y is greater than X. The justification of skewness preference

relating is often characterized by arguments invoking a "*ceteris paribus*" condition, which is intended to separate out the effect of higher order moments (and thus to focus solely upon the utility differences resulting from changes in skewness alone). Cain et al (2002) and Cain and Peel (2005) illustrate the logical fallacy in the context of simple mixed gambles in non-CPT models.

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[ii] National Premium bonds are tax-free and the expected return was recently raised to 3.2 per cent. A higher-rate taxpayer would have to earn a gross rate of around 5.33 per cent to exceed this and a standard rate taxpayer around 4%. The safe rate of interest always appears higher than the rate for a standard rate taxpayer and for many periods also for a higher ratepayer. Premium bonds have multiple prizes (like a lottery ticket with N prizes). The chances of winning any prize are 24,000 to 1 since 1 September 2004. The top prize is one million. The minimum purchase is £100

[iii] Tversky and Kahneman (1992) assumed the value function was of the power form with the same exponent over gains and losses. This specification is not appropriate for analyzing mixed gambles. For equal exponents stake size is indeterminate. With appropriately different exponents the agent becomes infinitely gain loving rather than loss averse over small (and also optimal stake gambles) violating the loss aversion assumption in CPT.