

## A result similar to the Odlyzko's "Paris Metro Pricing"

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# A Result Similar to the Odlyzko's "Paris Metro Pricing".

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## Abstract

We investigate the two-stage competition in which two Internet Service Providers choose sequentially their capacities and then their prices while facing a flow of new customers who decide to belong to one ISP or the other on the basis of a comparison of access prices and of expected congestion rates. At the equilibrium of the game a vertical differentiation between the Internet Service Providers endogenously emerges: the firm which provides the larger network has the lowest rate of congestion and the highest access price. The I.S.P providing the smallest network (thus the most congested) earns the larger profit. It will be noticed that the spontaneous functioning of oligopolistic competition produces a result similar to the Odlyzko's "Paris Metro Pricing": at the equilibrium the two competitors propose different prices and rates of congestion, the most expensive one being also the least congested

L12, L13

## 1 Introduction

In this paper we investigate the static competition in prices and capacities between I.S.P which face a flow of new customers who decide to belong to one I.S.P or the other on the basis of a comparison of access prices and of congestion rates that they are supposed to anticipate in a rational way.

The interesting specificity of the approach presented here is that the level of quality of the service provided by an ISP depends not only on the capacity of its network (this is the common point with the model of Gabszewicz and Thisse) but also on the number of its customers which are connected, i.e. on the demand which is addressed to him/her. Since the latter, as we shall see below, is itself a function of the rate of congestion, there is a circular process which it will be necessary to analyze precisely

The main result result of this paper is the existence of two "mirror" equilibria. In each equilibria the firm with the larger capacity sets the higher prices and is less congested than its competitor. Duopolistic equilibrium then reproduces the properties of the "Paris Metro Pricing" proposal of Odlyzko: consumer with different degrees of aversion to congestion are offered different (sub) networks: the more congestion adverse are allowed to pay more in order to benefit from a lower congestion rate.

## 1.1 The Model

There exist on the market two I.S.P each choosing a couple (price, quality),  $(p_1, q_1)$  for firm 1 and  $(p_2, q_2)$  for firm 2. The price is a lump sum which gives the right to the consumer to connect himself (herself) to the network of the ISP during a given period of time. The quality  $q_i$  of firm  $i$ 's service is supposed equal to the ratio  $\frac{k_i}{D_i}$  ( $\frac{D_i}{k_i}$  being the congestion rate of firm  $i$ 's network) where  $K_i$  represents the capacity of firm  $i$ 's network and  $D_i$  is the demand to firm  $i$  ( number of users).. This assumption is selected for the sake of simplicity. The basic idea is that the users always prefer a lower rate of congestion because the congestion present on a network represents a cost (measured in units of time) for the users since it delays or slows down the sending and the reception of data. This time increases with the number of connected users and decreases with the capacity of the network.

Consumers are characterized by their type  $\theta$  which measures their preference for quality (the higher is  $\theta$ , the more important is the preference for quality). They subscribe to one of both firms. We make the simplifying assumption according to which  $\theta$  is distributed uniformly on the interval  $[ 0,1 ]$ . The surplus of a type  $\theta$ -consumer subscribing to firm  $i$  may be written as :

$$S_i(\theta) = u + \theta q_i - p_i$$

where  $u$  is a strictly positive parameter, the value of which is supposed very high so that at the equilibrium every user prefers to subscribe to one of the firms rather than staying out (i.e. the market is always covered).

T

1. We shall study the perfect subgame Nash equilibria, of the following three-stage game:
2. in a first stage the ISPs choose in a simultaneous way the capacity of their respective networks, the firm  $i$  investing an amount of money  $C(K_i)$  in order to reach a capacity  $K_i$ . The function  $c$  is assumed increasing, convex and such that  $C(0) = 0$ . The capacity choices are irreversible and observable.
3. in a second stage, given the capacities previously selected, the ISPs choose, who rationally anticipate the equilibrium demand functions  $D_i(p_1, p_2)$ ,  $i = 1, 2$ , determined in the third stage, simultaneously choose their prices;
4. in the third stage each user chooses the network to which it will be connected given to the couple of prices  $(p_1, p_2)$  chosen previously by the ISPs and to the couple of rates of congestion which she/he is assumed to rationally anticipate.

## 1.2 The perfect subgame equilibrium

As usual the model is solved backward starting with the last stage, i.e. the determination of the demand functions. The type  $\theta^*$ -individual is indifferent between the two offers. All the consumers of types  $\theta > \theta^*$  contract with the high quality firm while the consumers of types  $\theta < \theta^*$  choose the low quality firm.

$$\theta^* q_1 - p_1 = \theta^* q_2 - p_2 \Rightarrow \theta^* = \frac{p_2 - p_1}{q_2 - q_1}.$$

The demand addressed to the low quality firm is equal to  $\theta^*$ . The demand addressed to the high quality firm is equal to  $(1 - \theta^*)$ . Straightforwardly the demand addressed to each firm depends on the expected rate of congestion. The realized rate of congestion is itself a function of the demand. The assumption of rational anticipations imposes an equality between both expected and realized rates of congestion for each I.S.P.

Let us first consider the case where firm 2 is the high quality firm ( $p_2 > p_1$ ):

The demand addressed to firm 2 is thus :

$$D_2 = 1 - \frac{p_2 - p_1}{q_2 - q_1} \quad (1)$$

where

$$q_2 = \frac{K_2}{D_2} \quad (2)$$

In the same way for firm 1

$$D_1 = \frac{p_2 - p_1}{q_2 - q_1}$$

and

$$q_1 = \frac{K_1}{D_1}$$

It follows that the difference between the "realized" qualities  $q_2 - q_1$  is a function of the difference between the expected qualities  $q_2^a - q_1^a$  :

$$q_2 - q_1 = \frac{K_2(q_2^a - q_1^a)}{(q_2^a - q_1^a)(p_2 - p_1)} - \frac{K(q_2^a - q_1^a)}{(p_2 - p_1)} \quad (3)$$

Under rational expectations  $q_2^a - q_1^a = q_2 - q_1$  and we determine the equilibrium value of  $q_2 - q_1$  as a function of both access prices and networks capacities:

$$q_2 - q_1 = \frac{(p_2 - p_1) [K_2 + K_1 + p_2 - p_1]}{(p_2 - p_1) + K_1} \quad (4)$$

In the symmetrical case where  $p_1 > p_2$  one obtains obviously a symmetrical formula

$$q_1 - q_2 = \frac{(p_1 - p_2) [K_2 + K_1 + p_1 - p_2]}{(p_1 - p_2) + K_2} \quad (5)$$

When  $p_1 \rightarrow p_2$ ,  $q_1 \rightarrow q_2$ .

This is the most interesting aspect of the model: a positive difference



in price generates a difference in quality so that the most expensive network is always the least congested..

By replacing  $q_2 - q_1$  by its equilibrium value one obtains the equilibrium demands for  $p_2 > p_1$ :

$$D_2 = 1 - \frac{(K_1 + (p_2 - p_1))}{(K_1 + K_2 + (p_2 - p_1))} \quad (6)$$

$$D_1 = \frac{(K_1 + (p_2 - p_1))}{(K_1 + K_2 + (p_2 - p_1))}$$

When  $p_1 > p_2$  the symmetrical formulas are:

$$D_1 = 1 - \frac{(K_2 + (p_1 - p_2))}{(K_1 + K_2 + (p_1 - p_2))} \quad (7)$$

$$D_2 = \frac{(K_2 + (p_1 - p_2))}{(K_1 + K_2 + (p_1 - p_2))}$$

Finally when  $p_2 = p_1$  we obtain  $D_i = \frac{K_i}{K_1 + K_2}$ ,  $i = 1, 2$ .

It is now possible to analyze the second stage equilibrium of the game. There is at this level of the play a Nash equilibrium in price between the Internet Service Providers. When  $p_2 > p_1$ , the respective profits of both firms are:

$$\begin{aligned}\pi_2 &= \left[1 - \frac{(K_1 + (p_2 - p_1))}{(K_1 + K_2 + (p_2 - p_1))}\right]p_2 - C(K_2) \\ \pi_1 &= \left[\frac{(K_1 + (p_2 - p_1))}{(K_1 + K_2 + (p_2 - p_1))}\right]p_1 - C(K_1)\end{aligned}\quad (8)$$

We obtain symmetrical formulas for  $p_1 > p_2$  :

$$\begin{aligned}\pi_1 &= \left[1 - \frac{(K_2 + (p_1 - p_2))}{(K_1 + K_2 + (p_1 - p_2))}\right]p_1 - C(K_1) \\ \pi_2 &= \left[\frac{(K_2 + (p_1 - p_2))}{(K_1 + K_2 + (p_1 - p_2))}\right]p_2 - C(K_2)\end{aligned}\quad (9)$$

When  $p_2 = p_1$  we obtain  $\pi_i = \frac{K_i}{K_1 + K_2}p_i - C(K_i)$ .

For the sake of simplicity we shall now assume that  $C(K_i) = 0.5 * K_i^2$ .

Deriving  $\pi_2$  with respect to  $p_2$  one obtains

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_2} &= \frac{K_2(K_1 + K_2 - p_1)}{(K_1 + K_2 - p_1 + p_2)^2} \text{ for } p_2 > p_1 \\ \frac{\partial \pi_2}{\partial p_2} &= \frac{K_2^2 + (p_2 - p_1)^2 + K_1(p_1 - 2p_2) + K_2(K_1 - 2p_2 + 2p_1)}{(K_1 + K_2 - p_2 + p_1)^2} \text{ for } p_1 > p_2\end{aligned}\quad (10)$$

Let us notice moreover that, for  $p_1 > p_2$  we have  $\frac{\partial^2 \pi_2}{\partial p_2^2} = -\frac{2K_1(K_1 + K_2 + p_1)}{(K_1 + K_2 - p_2 + p_1)^3} < 0$ .

**Lemma 1:**

1. The best reply correspondence of firm 2 is:

(i) when  $K_1 \geq K_2$  :

$$p_2 = +\infty \quad \text{for every } p_1 \in [0, K_2(1 + \frac{K_2}{K_1})]$$

$$p_2 = K_1 + K_2 + p_1 - \sqrt{K_1} \sqrt{K_1 + K_2 + p_1} \quad \text{for every } p_1 \geq K_2(1 + \frac{K_2}{K_1})$$

(ii) when  $K_2 \geq K_1$  :

$$p_2 = +\infty \quad \text{for every } p_1 \in [0, K_1 + K_2)$$

$$p_2 \in [p_1, +\infty] \quad \text{for } p_1 = K_1 + K_2$$

$$p_2 = p_1 \quad \text{for } p_1 \in [K_1 + K_2, K_2(1 + \frac{K_2}{K_1})]$$

$$p_2 = K_1 + K_2 + p_1 - \sqrt{K_1} \sqrt{K_1 + K_2 + p_1} \quad \text{for every } p_1 \geq K_2(1 + \frac{K_2}{K_1}).$$

**2.** The best reply correspondence of firm 1 is as follows:

(i) when  $K_2 \geq K_1$  :

$$p_1 = +\infty \quad \text{for every } p_2 \in [0, K_1(1 + \frac{K_1}{K_2})]$$

$$p_1 = K_1 + K_2 + p_2 - \sqrt{K_2} \sqrt{K_1 + K_2 + p_2} \quad \text{for every } p_2 \geq K_1(1 + \frac{K_1}{K_2})$$

(ii) when  $K_1 \geq K_2$  :

$$p_1 = +\infty \quad \text{for every } p_2 \in [0, K_1 + K_2)$$

$$p_1 \in [p_2, +\infty] \quad \text{for } p_2 = K_1 + K_2$$

$$p_1 = p_2 \quad \text{for every } p_2 \in [K_1 + K_2, K_1(1 + \frac{K_1}{K_2})]$$

$$p_1 = K_1 + K_2 + p_2 - \sqrt{K_2} \sqrt{K_1 + K_2 + p_2} \quad \text{for every } p_2 \geq K_1(1 + \frac{K_1}{K_2}).$$

**Lemma 2:** For any couple of networks capacities  $(K_1, K_2) \in \mathfrak{R}^{2+}$

there exists a unique second-stage Nash equilibrium between the Internet Access Providers which is such that:

(i) if  $K_2 > K_1$  then  $p_1 = K_1 + K_2$  and  $p_2 = \frac{1}{2} \left( K_2 + \sqrt{5K_2^2 + 4K_1K_2} \right)$

(ii) if  $K_2 < K_1$  then  $p_2 = K_1 + K_2$  and  $p_1 = \frac{1}{2} \left( K_1 + \sqrt{5K_1^2 + 4K_1K_2} \right)$

(iii) if  $K_2 = K_1$  then  $p_1 = p_2 = K_1 + K_2$ .

Taking into account Lemma 2, the profits of firm 2 are

$$\begin{aligned} \pi_2 &= K_2 - \frac{1}{2}K_2^2 \quad \text{if } K_2 \geq K_1 \\ \pi_2 &= \frac{(K_1 + K_2)(\sqrt{K_1(5K_1 + 4K_2)} - K_1)}{K_1 + \sqrt{K_1(5K_1 + 4K_2)}} - \frac{1}{2}K_2^2 \quad \text{if } K_2 \leq K_1 \end{aligned} \quad (11)$$

and symmetrically for firm 1.

**Lemma 3:**

1. The reaction function of firm 2 is as follows:

(i)  $K_2 = 1$  if  $K_1 \leq K^*$  where  $K^* \simeq 0.829$

(ii)  $K_2 = f(K_1)$  where  $f$  is a decreasing function such as  $f(0) = 1$

(see Figure below<sup>1</sup>)

2. The reaction function of firm 1 is exactly symmetrical.

**Proof:** For a given  $K_1$  there are two values of  $K_2$  corresponding potentially to a local maximum of the profit  $\pi_2$  :  $K_2 = 1$  et  $K_2 = f(K_1)$ .  $K_2 = 1$  can be a solution only for  $K_1 \leq 1$  In the same way  $K_2 = f(K_1)$  can be a solution only for  $f(K_1) \leq K_1$ , i.e. for  $K_1$  lower than  $\simeq 0.666$ . Consequently for values of  $K_1$  ranging between  $\simeq 0.666$  and 1 it is necessary to compare the profits respectively associated to

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<sup>1</sup>

The precise formula is too long and too complicated to be reproduced. The graph is plotted using *Mathematica*.

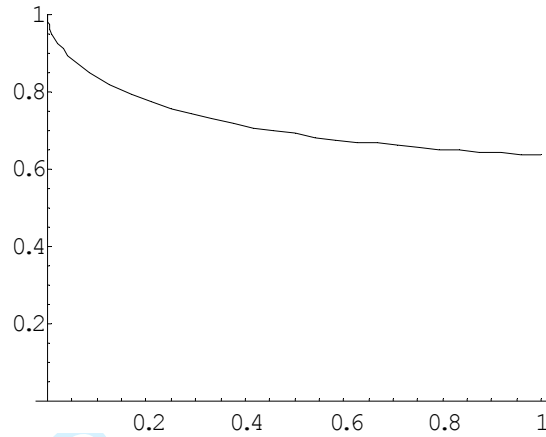


Figure 1:

$K_2 = 1$  and  $K_2 = f(K_1)$ . It results from this (cf Figure below for illustration) that  $K_2 = 1$  is a best reply if and only if  $K_1$  is lower than approximately 0.829. ■

The Figure below represents, according to  $K_1$ , the profits obtained by firm 2 when respectively it chooses  $K_2 = 1$  and  $K_2 = f(K_1)$

It follows from the preceding Lemma that the reaction function of each firm is discontinuous, constant (capacity equalizes 1) between 0 and approximately 0.829 then decreasing after 0.829 where the best reply jumps down from 1 to approximately 0.649. We deduce easily now the equilibrium of the game.

**Proposition 1** *There are two perfect subgame equilibria of the model.*

*These equilibria are symmetrical and such that:*

$K_2^* = 1, K_1^* \simeq 0.635909283988116769, p_1^* = 1.63590928398811685,$   
 $p_2^* = 1.87328412354767888$  (*equilibrium 1*) and

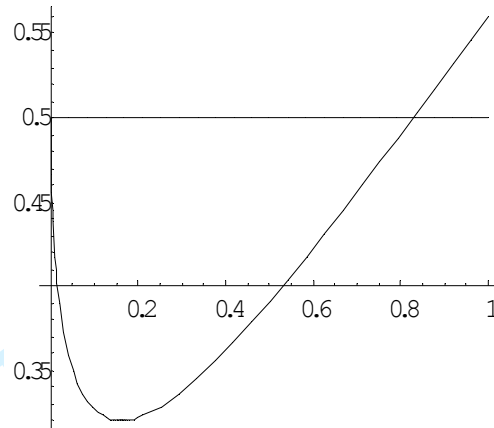


Figure 2:

$$K_1^* = 1, K_2^* \simeq 0.635909283988116769, p_2^* = 1.63590928398811685, \\ p_1^* = 1.87328412354767888 \text{ (equilibrium 2).}$$

**Proof:** See previous Lemmas.

There is thus at the equilibrium a vertical differentiation which is established in an endogenous way between the Internet Service Providers: the firm which provides the larger network has the lowest rate of congestion and the highest access price ... It is interesting to notice that the corresponding equilibrium profits are 0.560435 for the I.S.P providing the smallest network (thus the most congested) and 0.5 for the I.S.P providing the largest network (thus the least congested). There is in the game studied here a first-mover' advantage: since there are two symmetrical equilibria each Provider prefers being the one which provides the smallest network and makes the highest profit.

## References

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