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Pricing Inflation Linked Bonds

Paolo Falbo¹, Francesco M. Paris (1960-2005)¹, Cristian Pelizzari¹,²

Abstract

This paper advances a pricing model for inflation linked bonds. Our proposal is developed starting from a Vasicek model of the instantaneous inflation rate process (Vasicek (1977)) and the Cox, Ingersoll, and Ross (CIR) model for the nominal instantaneous risk-free interest rate process (Cox, Ingersoll Jr. and Ross (1985)). Instead of adopting the standard approach of a cross-section estimation of the term structure of real interest rates, this work proposes a pricing model based on the estimation of inflation risk premium. The model is applied to Treasury Inflation Protected Securities (TIPS’s), which are inflation linked bonds issued by the U. S. Department of the Treasury. Empirical validation is carried out on data in the period 1999-2005.

JEL classification: C14; C15; G12; G13.

Keywords: inflation linked bonds; continuous time stochastic models; interest rates; inflation rates; Treasury Inflation Protected Securities.

¹We are grateful to Prof. Paris for his original idea about this work. His example of strength before untold suffering is evidence that everything in our life is pure gift.

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1 Introduction

This paper deals with the problem of pricing inflation linked bonds. Such valuation has attracted the interest of the scientific community since a relatively short period, even though the first issues of these securities date back quite a long time ago. French and Finnish governments first issued these securities after World War II to stabilize their respective economies. They have been followed during the fifties and sixties by Argentina, Brazil, and Mexico. The United Kingdom started issuing inflation linked bonds since 1981 and they represent now nearly 26% of the British public debt. In the United States, emissions of inflation linked bonds started in 1997, with the first issues of TIPS’s (Treasury Inflation Protected Securities). Euro-zone countries have focused on this asset class more recently. The Italian Treasury first issued BTP-i’s (Buoni del Tesoro Poliennali indicizzati all’inflazione dell’Area Euro) in September 2003, with a constant increase of the principal amount issued every year.

Correct pricing of inflation linked bonds is a relevant problem since they receive a large interest form different agents. Investors buy them to defend the real value of an investment from inflation. They also benefit of low liquidity costs since these bonds are usually issued in large lots. Governments, on the other hand, can extend the range of their debt offer avoiding substantial increase of their risk. By issuing inflation linked securities, governments do not increase significantly their risk exposure since a rise in the inflation reduces the burden of the national debt in real terms. If these securities help reduce the cost of borrowing for national governments is on the contrary an open question, as empirical evidence is not uniform (e.g., Roll (1996) and Hunter and Simon (2005)).

The difficulties linked to the pricing of inflation-indexed bonds are due to their financial structure, which is not elementary. Usually these securities are composed by several coupon payments and a maturity capital refund. The capital value is re-valuated according to the
observed inflation rate. The owner receives a periodic payment (typically every six months) calculated applying a fixed rate to the re-valuated face value. At maturity an inflation linked bond pays the maximum between the re-valuated and the initial face value. The standard valuation problem of bonds is therefore complicated here by two additional elements: the risk of inflation and the valuation of financial options. The former is definitely more relevant than the latter, since the optional value of the final payment is usually close to its intrinsic value. Inflation risk (or, equivalently, the risk of real interest rate) is therefore the core of the research in this area.

We propose here a model to price inflation linked bonds explicitly based on the estimation of inflation risk premia. We assume that nominal interest rates follow a standard CIR model Cox et al. (1985), while we assume a Vasicek model Vasicek (1977) for the inflation rate. Such a starting point differs our work from previous related literature and is motivated by the economic reason that inflation rate can become negative. Although in real economies deflation is a rare event, nevertheless it is theoretically possible and, from an empirical point of view, it persistently appeared in Japan in the period 1999-2006 as a consequence (among other factors) of bank insolvency after the fall in prices of real estates.

Two papers are mostly related to our work. Jarrow and Yildirim (2003) develop a model to price TIPS’s and related derivative securities, starting from a three factor Gaussian HJM model Heath, Jarrow and Morton (1992) and adopting a two-step estimation procedure of the term structure of nominal and real forward rates. The authors obtain a closed formula to price index-linked derivatives neglecting risk premia parameters. However to get to such an interesting result they allow nominal interest rates to be normally distributed, which is a hardly sustainable hypothesis. Besides their estimation procedure forbids real forward rates from becoming negative. However negative interest rates are an economic outcome of periods when inflation is higher than money appreciation rate. Our approach does not imply any sign on real interest
rates and is therefore respectful of this possible state of the world. Indeed our model focuses on inflation risk premia and it does not require the estimation of the (unobservable) term structure of real rates. As a result we obtain evidence of an important economic variable and simplify the estimating procedure.

The second paper is that of Chen, Liu and Cheng (2005). In this case the authors develop and test a two factor *CIR* model, referred in particular to real interest rates and an inflation variable (mimicking the inflation rate). In a similar spirit to our work they also aim at estimating inflation risk premia, which result through a procedure combining a bootstrap and an unscented Kalman filter (*UKF*). Besides they estimate endogenously the correlation between real rates and inflation, which is a result not possible under the *HJM* model adopted by Jarrow and Yildirim (2003). The authors show a closed-form solution to both real and nominal zero coupon bond prices and obtain several insights about the way the inflation risk premia has evolved in the period 1998-2004 in the U.S.. A major drawback in this work is the adoption of a *CIR* model for the real rates and inflation factors. While such a choice allows an elegant treatment of equations, it implies again that those factors can not become negative. The plain Vasicek specification for inflation rates avoids such an invalidating problem, and balances a larger incidence of numerical calculations in our model.

We apply our model to American inflation linked bonds over the period 1999-2005. We find that inflation risk premia are often different from zero, passing from negative to positive (in 2001) for all the maturities analyzed here.

The paper is organized as follows. In the following section we briefly revise literature related to inflation rates and inflation linked securities valuation. Section 3 analyzes the pricing model, starting from the processes of nominal interest rate and inflation rate. Equivalent martingale change of measure is therein specified. Section 4 describes a "two-stage" methodology for estimating model parameters. In Section 5 an empirical analysis is performed on *TIPS*’s prices.
Section 6 concludes.

2 Related Literature

The work of Richard (1978) threw light on the links between real interest rate and inflation rate in a continuous time stochastic setting on the basis of the original contributions of Merton (1973) and Black and Scholes (1973). This model represents the archetype of bivariate models of interest rates and inflation rates; one drawback is that real interest rates cannot be observed in the absence of inflation linked securities.

Malliaris and Malliaris (1991) propose a bivariate model for the nominal interest rate and the inflation rate suggesting that they follow Ito processes. The authors deduce the dynamics of the real interest rate and propose a continuous time stochastic version of Fisher's equation (Fisher (1930)). A numerical example on U.S. data from 1865 to 1972 is presented; the proxies of inflation rate and nominal interest rate evolve according to an arithmetic Brownian motion, allowing for negative values of the nominal interest rate.

A group of works has focused on the estimation problem of the term structure of real interest rates. Woodward (1990) develops an empirical investigation on prices of British inflation linked bonds and obtains implicit market estimates of real interest rates and expected inflation. He assumes that inflation risk premium is zero. In a similar spirit, Roll (2004) observes TIPS's prices to estimate real yield curves, which are then compared with nominal yield curves to derive the term structure of anticipated inflation. Brown and Schaefer (1994) also estimate the term structure of real interest rates by extracting them from prices of British inflation linked bonds. The work of Hunter and Simon (2005) analyzes the impact of TIPS’s in diversified portfolios; the authors conclude that, in an efficient market, inflation risk does not add/remove any systematic risk. This result implies that inflation risk premium should not differ significantly from zero and so it reinforces the work of Woodward (1990).
An increasing interest of the scientific literature has recently directed to inflation linked
derivatives, a new generation of instruments dealing with inflation rates, namely swaps and
options. According to Kruse (2007), two groups of works face the problem of pricing of inflation
linked derivatives. The first group models the relation of nominal, real and inflation rates
adopting the original equation of Fisher (1930). These papers include Jarrow and Yildirim
(2003), which base their model on that of Heath et al. (1992) to price plain vanilla call options
on inflation index; Hughston (1998) also refers to Heath et al. (1992) model and obtains
analytical expressions for the price of inflation linked derivatives on the actual inflation. Finally,
Kjaergaard (2007) imposes a three-factor Gaussian model, as in Jarrow and Yildirim (2003), on
the shifts between the forward curves of nominal and real interest rates as well as the consumer
price index and their spot counterparts; the author derives closed form solutions for the price of
year-on-year inflation swaps and call options on inflation. Alternatively a second group of works
including Belgrade, Benhamou and Koehler (2004), and Mercurio (2005) model the forward
inflation index as a geometric Brownian motion, which is a martingale under its respective
forward measure. The Fisher equation is not necessarily preserved with these models, which have
in general more parameters to be estimated with respect to the first group. The first paper
establishes no arbitrage relationships between zero coupon and year-on-year swaps. The second
presents closed form formulas for pricing zero coupon swaps and year-on-year swaps and
contrasts the results of an application with those of Jarrow and Yildirim (2003).

Other recent contributions develop more sophisticated derivative models. Kruse (2007) extends a
previous work of Korn and Kruse (2004) to account for stochastic volatility of the inflation index
as in Heston (1993) and obtains closed form solutions to the price of inflation linked caplets on
model to account for stochastic volatility consistent with smile effects of option prices and derive
an analytical solution for inflation linked caps and floors.
3 Pricing Bonds Linked to Domestic Inflation: A Continuous Time Model

In this section, we will lay down a continuous time stochastic model for pricing Treasury Inflation Protected Securities.

TIPS’s pay coupons at a fixed coupon rate at regular intervals (generally every six months) and the principal at expiration. Both coupons and the principal are subject to variation according to a coefficient, \( CI \), measuring inflation accrued from the bond issuing date, \( g \), to the coupon or principal payment dates. While coupons increase in case of inflation and decrease in case of deflation, the redemption value is floored to the face value (\( FV \)). At expiration, \( H \), a TIPS has the following payoff:

\[
FV + \max \left( (CI_H \times FV - FV), 0 \right).
\]

To illustrate the coupon indexation mechanism, let us suppose that the next coupon expires at \( h \) and that the semiannual coupon rate is \( i_2 \) dollars per dollar of face value; thus, the inflation linked coupon value is \( i_2 \times CI_h \times FV \).

The indexing coefficient \( CI_h \) is computed according to the following formula:

\[
CI_h = \frac{IR_h}{IR_g} = 1 + \left( \frac{IR_h}{IR_g} - 1 \right) = 1 + I_h;
\]

\( I_h \) is the inflation rate of the period starting at \( g \), the issuing date, and ending at \( h \), date corresponding to the \( d \)th day of the \( m \)th month after the month of \( g \). The reference index, \( IR_h \), is defined as:

\[
IR_h = IE_{m-3} + \left( \frac{d - 1}{gg_m} \right) (IE_{m-2} - IE_{m-3});
\]  

(1)

\( IE_m \) represents the U.S. Consumer Price Index - All Urban Consumers (CPI-AUC) referred to month \( m \). \( IE \) is generally published 15 days after the end of every month by the U.S. Department of Labor - Bureau of Labor Statistics. Finally, \( gg_m \) is the number of days of month
Notice that \( I_h \) refers to the period between \( g \) and \( h \), but its formula takes into account inflation with a delay, as \( IR \) is based on a linear interpolation between inflation of the second and third preceding months. On the other hand, considering a publication date of inflation data by midmonth, the indexation coefficient can be calculated for at most the next month; indeed, \( CI \) for all the days of month \( m + 1 \) is computed by using \( IE_{m-1} \) and \( IE_{m-2} \), which are both known in the second half of month \( m \). In conclusion, the exact value of a coupon to be paid at the mid of month \( m \) (as usual for TIPS’s) is determined at most one month before its detachment (at the mid of month \( m - 1 \)).

In order to relay on well established mathematical techniques, the model we develop in this section is a continuous time model characterized by two fundamental variables: the instantaneous(ly compounded) inflation rate \( i(t) \) and the instantaneous(ly compounded) nominal risk-free interest rate \( r(t) \), with \( t \) representing current time of evaluation. We will see in Section 4 how to estimate \( i(t) \) and \( r(t) \) from market data.

We assume that \( i(t) \) and \( r(t) \) have the following stochastic dynamics:

\[
di(t) = \alpha \left[ \beta - i(t) \right] dt + \sigma_i dz_i(t), \quad \alpha, \sigma_i > 0 \tag{2}
\]

and

\[
dr(t) = a \left[ b - r(t) \right] dt + \sigma_r \sqrt{r(t)} dz_r(t), \quad a, b, \sigma_r > 0, \tag{3}
\]

respectively. Furthermore, we assume correlation between the two Brownian motions. In particular, it is:

\[
dz_i(t) dz_r(t) = \rho_{ir} dt. \tag{4}
\]

It is intended that the usual probabilistic assumptions about the economy hold, i.e. the existence of a probability space \( (\Omega, \mathcal{F}, P) \), with \( \Omega \) the space of elementary events, \( \mathcal{F} \) a \( \sigma \)-algebra

\footnote{Notice that, in our framework, \( r(t) \) has a zero credit risk, while it can be subject both to inflation and interest rate risks.}
constructed on $\Omega$, and $P$ the statistical (or "physical", or "real world") probability of events constructed on $(\Omega, \mathcal{F})$. Moreover, $\mathbb{F}_t$ is the standard filtration generated by the two Brownian motions $z_i(t)$ and $z_r(t)$, with $t \in [0, T]$.

With respect to the inflation rate, we adopt the same dynamics assumed by Vasicek (1977) for the local interest rate. The drawback of this process, allowing for negative values of the stochastic variable, is justified with inflation. In fact, negative values of the inflation rate could indicate deflation, while negative values of the nominal interest rate would be more difficult to explain. The nominal risk-free interest rate is then modeled with the dynamics proposed by Cox et al. (1985); this model main merit is that of not allowing for negative or zero values of the nominal interest rate.

Next we write down the PDE’s and their relative boundary conditions to be satisfied by each specific component of TIPS’s prices. We simplify the notation by introducing the following equalities: $\alpha [\beta - i(t)] = \Lambda_i$, $a [b - r(t)] = \Lambda_r$, $\sigma_i = \Sigma_i$, and $\sigma_r \sqrt{r(t)} = \Sigma_r$. Let us also suppose that the bond to be priced has a face value of 1 dollar.

For a general coupon $C_h$, expiring at $t = h$, the PDE and the connected boundary condition are:

$$
\frac{\partial C_h}{\partial t} + (\Lambda_i - \lambda_i \Sigma_i) \frac{\partial C_h}{\partial i} + (\Lambda_r - \lambda_r \Sigma_r) \frac{\partial C_h}{\partial r} + \frac{1}{2} \sum_i \frac{\partial^2 C_h}{\partial i^2} + \frac{1}{2} \sum_r \frac{\partial^2 C_h}{\partial r^2} + \frac{1}{2} \sum_i \sum_r \frac{\partial^2 C_h}{\partial i \partial r} \rho_{ir} + \frac{1}{2} \sum_i \sum_r \frac{\partial^2 C_h}{\partial r \partial i} \rho_{ri} = r C_h
$$

$$
C_{h,h}(i) = i_2 CI_h.
$$

where $i_2$ is the coupon rate (paid semiannually), $t$ is the valuation time, and $CI_h$ is the indexing coefficient at time $h$ of expiration.

With respect to principal appreciation, $R_H$, to be paid at bond maturity in case of inflation, the
PDE and the boundary condition are:

\[
\frac{\partial R_H}{\partial t} + (\Lambda_i - \lambda_i \Sigma_i) \frac{\partial R_H}{\partial i} + (\Lambda_r - \lambda_r \Sigma_r) \frac{\partial R_H}{\partial r} + \\
\frac{1}{2} \Sigma_i^2 \frac{\partial^2 R_H}{\partial i^2} + \frac{1}{2} \Sigma_r^2 \frac{\partial^2 R_H}{\partial r^2} + \frac{1}{2} \Sigma_i \Sigma_r \frac{\partial^2 R_H}{\partial i \partial r} \rho_{ir} + \frac{1}{2} \Sigma_i \Sigma_r \frac{\partial^2 R_H}{\partial r \partial i} \rho_{ri} = r R_H
\]

\[
R_{H,H}(i) = \max [(CI_H - 1), 0].
\]

Finally, the bond face value, which depends on the instantaneous nominal interest rate but not on the inflation rate, is characterized by a simpler form of both the PDE and the boundary condition:

\[
\frac{\partial P_H}{\partial t} + (\Lambda_r - \lambda_r \Sigma_r) \frac{\partial P_H}{\partial r} + \frac{1}{2} \Sigma_r^2 \frac{\partial^2 P_H}{\partial r^2} = r P_H;
\]

\[
P_{H,H}(r) = 1.
\]

Let us turn, now, to the pricing problem. We want to look at each single element of the TIPS’s price and analyze further its structure. Suppose that bond pricing is performed at time \( t \), with \( g \leq t < h \). Thus, the value at expiration of a general coupon \( C_h \) is expressed by the following stochastic quantity:

\[
C_{h,h} = i_2 e^{S_i(g,h)}.
\]

The appropriate discounting of Eq. (5) leads to the current value of the coupon; we have:

\[
C_{h,t} = i_2 E_t \left( e^{S_i(g,h)} e^{-S_i(t,h)} \right).
\]

In the previous formula, \( E_t \) is the expectation operator taken at time \( t \), while \( S_r(t,h) \) and \( S_i(g,h) \) have the following expressions:

\[
S_r(t,h) = \int_t^h r(u) \, du + \frac{1}{2} \int_t^h \lambda_r^2(u) \, du + \int_t^h \lambda_r(u) \, dZ_r(u), \ t < h,
\]

with \( \lambda_r \) representing the interest risk premium, and

\[
S_i(g,h) = \int_g^h i(u) \, du + \frac{1}{2} \int_g^h \lambda_i^2(u) \, du + \int_g^h \lambda_i(u) \, dZ_i(u), \ g < h,
\]
where \( \lambda_i \) is the inflation risk premium. \( e^{S_{i}(g,h)} \) represents the stochastic compounding factor from \( g \) to \( h \), while \( e^{-S_{r}(t,h)} \) represents the stochastic discount factor from the coupon maturity \( h \) back to current time \( t \).

In a similar fashion, we can write the current value of principal appreciation due to inflation. If \( t \) is such that \( g \leq t < H \), the current value of principal appreciation is given by:

\[
R_{H,t} = E_t \left\{ \max \left[ \left( e^{S_{i}(g,H)} - 1 \right), 0 \right] e^{-S_{r}(t,H)} \right\}.
\]  (9)

It is evident from Eq. (9) that principal appreciation can be interpreted as a European style contingent claim with strike price of 1 dollar.

The last component to be priced is the TIPS’s face value. The value of 1 dollar of principal to be repaid at maturity \( H \) is worth in \( t \):

\[
P_{H,t} = E_t \left( e^{-S_{r}(t,H)} \right).
\]  (10)

Based on Eq. (6), (9), and (10), the current theoretical value of a TIPS is:

\[
TIPS_t = \left( \sum_{h>t}^{H} C_{h,t} \right) + R_{H,t} + P_{H,t}.
\]  (11)

The price defined in Eq. (11) can be explicitly computed according to the assumptions made with respect to the stochastic processes governing the instantaneous inflation and interest rate.

In the next subsections, we show how the three components of Eq. (11) can be analytically formulated, after a "change of measure" has been performed on the bivariate model of Eq. (2), (3), and (4).

### 3.1 Inflation and Interest Rates under the "Risk-Neutral" Dynamics

It is now convenient to assume the absence of arbitrage opportunities and write the inflation and interest rate dynamics under a martingale measure \( Q \) equivalent to the statistical probability \( P \).
The drift-adjusted processes are:

\[
d\tilde{t}(t) = \{\alpha [\beta - \tilde{i}(t)] - \sigma_i (\rho_{it} \lambda_i + \sqrt{1 - \rho_{it}^2} \lambda_i)\} dt + \sigma_i dz^Q_i(t) \\
= \alpha [\beta - \tilde{i}(t)] dt + \sigma_i dz^Q_i(t), \tag{12}
\]

and

\[
d\tilde{r}(t) = \{a [b - \tilde{r}(t)] - \lambda_r \sigma_r \sqrt{\tilde{r}(t)}\} dt + \sigma_r \sqrt{\tilde{r}(t)} dz^Q_r(t) \\
= \tilde{a} [b - \tilde{r}(t)] dt + \sigma_r \sqrt{\tilde{r}(t)} dz^Q_r(t), \tag{13}
\]

where

- \( z^Q_i(t) \) and \( z^Q_r(t) \) represent standard Brownian motions defined under the equivalent martingale measure \( Q^4 \),
- \( \lambda_i = \pi_i \), with \( \pi_i \in \mathbb{R} \), is the inflation risk premium,
- \( \lambda_r = \frac{\pi_r \sqrt{\tilde{r}(t)}}{\sigma_r} \), with \( \pi_r \in \mathbb{R} \), is the interest risk premium,
- \( \tilde{\beta} = \beta - \frac{\sigma_i (\rho_{it} \lambda_i + \sqrt{1 - \rho_{it}^2} \lambda_i)}{\alpha} \),
- \( \tilde{a} = a + \pi_r \),
- \( \tilde{b} = \frac{ba}{a + \pi_r} \).

The functional forms of \( \lambda_i \) and \( \lambda_r \) are taken from the original works of Vasicek (1977) and Cox et al. (1985), respectively, and define the martingale measure \( Q \) equivalent to probability \( P \).

The change of measure from \( P \) to \( Q \) has an important effect in terms of pricing formulas.

Indeed, under \( Q \), Eq. (7) and (8) become

\[
\hat{S}_r(t, h) = \int_t^h \tilde{r}(u) du, \ t < h,
\]

\footnote{Correlation between the new Brownian motions is equal to the old one, i.e. \( dz^Q_i(t)dz^Q_r(t) = \rho_{ir} dt \), as clearly pointed out by Joshi (2003), page 248.}
and

$$\hat{S}_i (g, h) = \int_g^h i (u) \, du, \ g < h,$$

respectively. Provided with these results, we want to show in the next subsections how Eq. (6), (9), and (10) change aspect and become more manageable.

### 3.2 Coupons

After change of measure, Eq. (6) becomes as follows:

$$C_{h,t} = i_2 \hat{E}_t \left( e^{\hat{S}_i (g, h)} e^{-\hat{S}_r (t, h)} \right),$$

where $\hat{E}_t$ is the expectation operator taken under the equivalent martingale measure $Q$.

It is important to recall the schedule of indexing coefficient $CI$ release. According to Eq. (1) and if time $t$ of evaluation falls in month $m$ after the issue date $g$, we know that indexing coefficients can be revealed up to the last day of month $m + 1$. Therefore, the above expectation can be simplified by observing that inflation between $g$ and $t'$, with $t' \geq t$, is already known\(^5\); indeed,

$$C_{h,t} = i_2 \hat{E}_t \left( e^{\hat{S}_i (g, t')} e^{-\hat{S}_r (t, h)} \right) = i_2 e^{\hat{S}_i (g, t')} \hat{E}_t \left( e^{\hat{S}_i (t', h)} e^{-\hat{S}_r (t, h)} \right), \quad (14)$$

and the constant $e^{\hat{S}_i (g, t')}$ is represented by $CI_{t'}$, the indexing coefficient calculated at time $t'$, with $t' \geq t$.

The expected value in Eq. (14) can be expressed through an alternative way by relaying on the well known formula for the covariance between two random variables $X$ and $Y$:

$$\text{cov} (X, Y) = \text{corr} (X, Y) \sigma (X) \sigma (Y) = \mathbb{E}(XY) - \mathbb{E}(X) \mathbb{E}(Y).$$

\(^5\)If $t$ falls in the first half of month $m$, generally $IE_{m-1}$ has not been published yet; therefore, $t'$ represents the end of month $m$. If $t$ falls in the second half of month $m$, generally $IE_{m-1}$ is known and $t'$ represents the end of month $m + 1.$
From such a formula, assuming $\text{corr}_t(e^{\tilde{S}_t(t,h)}, e^{-\tilde{S}_r(t,h)}) = k_t$, we get:

$$
\hat{E}_t \left( e^{\tilde{S}_t(t,h)} e^{-\tilde{S}_r(t,h)} \right) = \hat{E}_t \left( e^{-\tilde{S}_r(t,h)} \right) \hat{E}_t \left( e^{\tilde{S}_r(t,h)} \right) + k_t \hat{\sigma}_t \left( e^{-\tilde{S}_r(t,h)} \right) \hat{\sigma}_t \left( e^{\tilde{S}_t(t,h)} \right).
$$

(15)

The first expected value in the r.h.s. of Eq. (15), $\hat{E}_t \left( e^{-\tilde{S}_r(t,h)} \right) = \hat{E}_t \left( e^{-\int_t^h \tilde{r}(u) du} \right)$, represents the time $t$ risk-neutral value of a zero coupon bond paying 1 dollar at time $h$. Within the Cox et al. (1985) framework, such a value is given in closed form as follows:

$$
\hat{E}_t \left( e^{-\int_t^h \tilde{r}(u) du} \right) = e^{-A(t,h)\tilde{r}(t)+B(t,h)},
$$

(16)

with

$$
A(t, h) = \frac{2 \left( e^{\gamma(h-t)} - 1 \right)}{(\gamma + \bar{a}) \left( e^{(h-t)} - 1 \right) + 2\gamma},
$$

(17)

$$
B(t, h) = \frac{2\bar{a}b}{\sigma_r^2} \ln \left( \frac{2\gamma e^{(\gamma + \bar{a})(h-t)} - 2\gamma}{(\gamma + \bar{a}) \left( e^{(h-t)} - 1 \right) + 2\gamma} \right),
$$

(18)

and

$$
\gamma = \sqrt{\bar{a}^2 + 2\sigma_r^2}.
$$

(19)

The first standard deviation in the r.h.s. of Eq. (15) can be written:

$$
\hat{\sigma}_t \left( e^{-\tilde{S}_r(t,h)} \right) = \left\{ \hat{E}_t \left[ \left( e^{-\int_t^h \tilde{r}(u) du} \right)^2 \right] - \hat{E}_t^2 \left( e^{-\int_t^h \tilde{r}(u) du} \right) \right\}^{\frac{1}{2}}
$$

$$
= \left\[ \hat{E}_t \left( e^{-\int_t^h 2\tilde{r}(u) du} \right) - \hat{E}_t^2 \left( e^{-\int_t^h \tilde{r}(u) du} \right) \right\]^{\frac{1}{2}};
$$

the second expected value of the previous formula is given by Eq. (16) squared, while the first expected value is again Eq. (16), with $\bar{b}$ replaced by $2\bar{b}$, $\sigma_r$ replaced by $\sqrt{2}\sigma_r$, and $\tilde{r}(t)$ replaced by $2\tilde{r}(t)$, respectively.\(^6\)

\(^6\)The result follows immediately from $d(2\tilde{r}(t)) = 2d\tilde{r}(t) = 2\tilde{b}[\tilde{b} - \tilde{r}(t)]dt + 2\sigma_r \sqrt{\tilde{r}(t)}dz^Q(t)$, which can also be written as $d(2\tilde{r}(t)) = \tilde{b}[\tilde{b} - \tilde{r}(t)]dt + \sigma_r \sqrt{\tilde{r}(t)}dz^Q(t)$. Substituting $2\tilde{r}(t)$ with $\tilde{R}(t)$, we get $d(\tilde{R}(t)) = \tilde{b}[\tilde{b} - \tilde{R}(t)]dt + \sigma_r \sqrt{\tilde{R}(t)}dz^Q(t)$, which represents a stochastic differential equation à la CIR. Results from estimation and pricing of stochastic zero coupon bonds based on $\tilde{r}(t)$ can be used in the case of $\tilde{R}(t)$ by making the substitutions mentioned in the main text.
As far as the second expected value and standard deviation in the r.h.s. of Eq. (15) are concerned, it can be shown that $e^{S_i(t',h)}$ is a lognormally distributed random variable whose first two moments are:

$$\mathbb{E}_t[e^{S_i(t',h)}] = e^{\left(\bar{m}_h + \frac{\bar{v}_h^2}{2}\right)}$$  and  $$\text{Var}_t[e^{S_i(t',h)}] = \left(e^{\bar{v}_h^2} - 1\right)e^{2\bar{m}_h + \bar{v}_h^2},$$

respectively; besides, $\bar{m}_h$ and $\bar{v}_h^2$ represent the expected value and variance of the normal random variable $\bar{S}_i(t,h)$; their expressions are

$$\bar{m}_h = \frac{1}{\alpha}(1 - e^{-\alpha(h-t')})i(t') + \alpha\tilde{\beta}\int_{t'}^{h}\int_{t'}^{u}e^{-\alpha(u-s)}dsdu$$

$$= \frac{1}{\alpha}(1 - e^{-\alpha(h-t')})(i(t') - \tilde{\beta}) + \tilde{\beta}(h - t')$$

and

$$\bar{v}_h^2 = \frac{\sigma^2}{2\alpha^2}\left((h-t') + \frac{1}{2\alpha}(1 - e^{-2\alpha(h-t')}) + \frac{2}{\alpha}(e^{-\alpha(h-t')} - 1)\right).$$

### 3.3 Principal Appreciation

According to Eq. (9), the present value of principal appreciation is given by:

$$R_{H,t} = \mathbb{E}_t\left\{\max\left(\left(e^{S_i(g,H)} - 1\right),0\right) e^{-S_r(t,H)}\right\},$$

which can be alternatively written, under the equivalent martingale measure $Q$, as:

$$R_{H,t} = \mathbb{E}_t\left\{\max\left(\left(e^{\bar{S}_i(g,H)} - 1\right),0\right) e^{-\bar{S}_r(t,H)}\right\}.$$  

Again, like the coupons, inflation from $g$ to $t'$ is already known; thus, the previous expression can be restated as

$$R_{H,t} = \mathbb{E}_t\left\{\max\left(\left(CI_{t'}e^{\bar{S}_i(t',H)} - 1\right),0\right) e^{-\bar{S}_r(t,H)}\right\},  \quad (20)$$

where $CI_{t'}$ is the indexing coefficient of time $t' \geq t$ (see comments before Eq. (14)).

Such an expected value can be calculated through Monte Carlo techniques, by drawing the two variables, $e^{\bar{S}_i(t',H)}$, which is lognormally distributed, and $e^{-\bar{S}_r(t,H)} = e^{-\int_{t'}^{t} \bar{r}(u)du}$. No density
function is known explicitly for the last stochastic variable. One natural way to deal with this problem is numerical approximation of \( \int_t^H \dot{r}(u) \, du \); it is possible to discretize time interval \([t, H]\) in, say, \( S \) subintervals of width \( \delta = \frac{H-t}{S} \), such that their bounds are, respectively, \( T_0 = t \), \( T_1 = t + \delta \), \( \ldots \), \( T_i = t + i\delta \), \( \ldots \), \( T_S = t + S\delta = H \). Now, the integral can be approximated by

\[
\int_t^H \dot{r}(u) \, du \approx \sum_{i=0}^{S-1} \dot{r}(T_i)(T_{i+1} - T_i),
\]

where values of \( \dot{r}(T_i) \), \( i = 0, \ldots, S - 1 \), are sampled from a non central chi-square random variable.

Indeed, as pointed out in Cox et al. (1985), the transition density of \( r(t) \) given \( r(u) \) is:

\[
r(t) = e^{\chi^2_{df}(nc)} \cdot t > u,
\]

where \( \chi^2_{df}(nc) \) stands for a chi-square random variable with parameter of non centrality \( nc \) and \( df \) degrees of freedom; moreover, \( e \) is a constant of proportionality. In particular,

\[
n = \frac{4\bar{a}e^{-\bar{a}(t-u)}}{\sigma_r^2(1-e^{-\bar{a}(t-u)})} r(u),
\]

\[
df = \frac{4\bar{b}\bar{a}}{\sigma_r^2},
\]

and

\[
e = \frac{\sigma_r^2(1-e^{-\bar{a}(t-u)})}{4\bar{a}}.
\]

Therefore, indicating the \( j \)-th simulation of \( X_t = e^{-\hat{S}_r(t,H)} = e^{-\int_t^H \dot{r}(u) \, du} \) with \( x_{t,j} \), we can write

\[
x_{t,j} = e^{-\sum_{i=0}^{S-1} \dot{r}_j(T_i)(T_{i+1} - T_i)}, \quad j = 1, \ldots, N.
\]

If the degrees of freedom \( df \) are greater than 1, an important simplification\(^7\) can be applied when drawing \( \dot{r}(T_i) \), given \( \dot{r}(T_{i-1}) \); indeed, in this case

\[
\chi^2_{df}(nc) \equiv (Z + \sqrt{nc})^2 + \chi^2_{df-1},
\]

indicating that the non-central chi-square random variable on the left has the same distribution of the sum of two independent random variables, precisely the square of a shifted standardized normal, \( (Z + \sqrt{nc})^2 \), and a central chi-square, \( \chi^2_{df-1} \), with \( df - 1 \) degrees of freedom.

\(^7\)See Glasserman (2003), pages 121-123.
3.4 Correlation, Coupons and Principal Appreciation

In the previous two subsections, we have proposed pricing formulas (Eq. (14) and Eq. (20)) for coupons and principal appreciation; coupons depend on the correlation, \( k_t \), between \( e^{\hat{S}_i(t',h)} \) and \( e^{-\hat{S}_r(t,h)} \), while principal appreciation requires (correlated) random draws of \( e^{\hat{S}_i(t',H)} \) and \( e^{-\hat{S}_r(t,H)} \). Eventually, we know that \( e^{\hat{S}_i(t',h)} \) is lognormally distributed, while \( e^{-\hat{S}_r(t,h)} \) is simulated via Monte Carlo methods.

When drawing jointly non-normal random variables with nonzero correlation, many solutions may be adopted. The literature on the subject is particularly developed for multivariate normal variables, while the case of joint non-normality soon becomes analytically unmanageable, especially when the marginal densities do not belong to the same family. Calculations of coupon values according to Eq. (14) show that \( k_t \) does not influence significantly the final result for two reasons: first, in Eq. (14) \( k_t \) multiplies the standard deviations of two variables \( \hat{\sigma}_t \left( e^{-\hat{S}_r(t,h)} \right) \) and \( \hat{\sigma}_t \left( e^{\hat{S}_i(t',h)} \right) \), which are small; second, \( k_t \) itself has a small value. As we will show in Section 5, estimated correlation between \( i(t) \) and \( r(t) \) is a small positive value; furthermore, this relationship is weakened by the transformations \( i(t) \) and \( r(t) \) undertake when plugged into \( e^{\hat{S}_i(t',h)} \) and \( e^{-\hat{S}_r(t,h)} \). Therefore, to keep the matter as simple as possible, we do maintain correlation between \( i(t) \) and \( r(t) \), while correlation between \( e^{\hat{S}_i(t',h)} \) and \( e^{-\hat{S}_r(t,h)} \) is set to 0. This assumption transforms Eq. (14) into

\[
\mathbf{E}_t \left( e^{\hat{S}_i(t',h)} e^{-\hat{S}_r(t,h)} \right) = \mathbf{E}_t \left( e^{-\hat{S}_r(t,h)} \right) \mathbf{E}_t \left( e^{\hat{S}_i(t',h)} \right).
\]

To price principal appreciation, random draws of \( e^{\hat{S}_i(t',H)} \) and \( e^{-\hat{S}_r(t,H)} \) can now be performed as if the two random variables were linearly independent, i.e. uncorrelated; though, uncorrelatedness does not exclude non-linear dependence. If we could make the stronger assumption of independence between \( e^{\hat{S}_i(t',H)} \) and \( e^{-\hat{S}_r(t,H)} \), we would "separately" draw from the two variables, with a considerable algorithmic complexity reduction. This requirement is
justified by the effect on principal appreciation value of different jointly drawn couples from the
two variables; indeed, different values of $k_t$ do not seem to produce a value of principal
appreciation distinct from the case of independent draws. The joint drawing has been performed
using an immediate and simple algorithm, which is based on rank (or Spearman) correlation
between two variables. Rank correlation is different from the widely adopted linear (or Pearson)
correlation, but has the important property of invariance to monotonic transformations (see
Fackler (1999)); based on rank correlation, one is able to "generate dependent random variables
with any marginal distributions" (Fackler (1999)). Interpreting $k_t$ as rank correlation between
$e^{\hat{S}_r(t,H)}$ and $e^{-\hat{S}_r(t,H)}$ allows us to apply Fackler’s "copula style" algorithm; nonetheless, letting
$k_t$ vary in $[-1,1]$ does produce values of $R_{H,t}$ in Eq. (20) similar to the case of independent
draws. Consequently, we are allowed to act as if $e^{\hat{S}_r(t,H)}$ and $e^{-\hat{S}_r(t,H)}$ were independent, and
not only linearly uncorrelated.

3.5 Principal

Finally, the current value of 1 dollar of face value is simply given by the stochastic discount
factor derived by Cox et al. (1985). From Eq. (16), we can write:

$$P_{H,t} = E_t\left(e^{-S_r(t,H)}\right) = \hat{E}_t\left(e^{-\hat{S}_r(t,H)}\right) = e^{-A(t,H)\hat{r}(t)+B(t,H)},$$

where bond maturity $H$ substitutes coupon maturity $h$, while $A(t, H)$ and $B(t, H)$ are computed
according to Eq. (17), (18), and (19).

4 Estimation of Model Parameters

We comment briefly on the parameters characterizing the model of Jarrow and Yildirim (2003).
The authors design a four-step piecewise constant term structure of real and nominal forward
interest rates by a "stripping" procedure involving TIPS’s and nominal Treasury bonds prices.
This "simple" term structure can not be further sophisticated, since the authors have five TIPS’s
prices series at hand and estimating more rates would not guarantee a solution to the minimization. Forward rates estimates are constrained to be nonnegative (even if real rates can well be negative). The authors obtain an estimate of the nominal and real forward term structures for each day in the period April 15, 1999 to July 31, 2001.

For the real forward rate model, the authors assume "a one-factor volatility function ... with an exponentially declining volatility of the form,

\[ \sigma_r(t, T) = \sigma_r e^{-a_r(T-t)}, \]

where \( \sigma_r, a_r \) are constants" to be estimated. A similar model of volatility is adopted for the nominal forward rates, which requires the estimation of other two parameters, \( \sigma_n \) and \( a_n \).

Finally the inflation index, \( I(t) \), is modeled as a geometric Brownian motion whose volatility term is proportional to a diffusion coefficient, \( \sigma_I \). In addition, three correlation coefficients estimates (between spot nominal, spot real and inflation rates) are required to accomplish an empirical application.

The three-factor model of Jarrow and Yildirim (2003) relies on two "heavy" assumptions in the estimation process: the specification of two term structures of forward rates (four-step piecewise constant and positive), and the specification of two deterministic volatility functions. In conclusion the model of Jarrow and Yildirim (2003) requires a total of eight coefficient estimates (\( \sigma_r, a_r, \sigma_n, a_n, \sigma_I \), plus three correlations), although it should be noted that the estimated four-step piecewise constant term structures of real forward rates raise this number.

The model of Jarrow and Yildirim (2003) avoids the estimation of state variables risk premia. However, if one was interested in analyzing also those quantities, three drift estimates are additionally required (\( \alpha_n(t, T), \alpha_r(t, T), \) and \( \mu_I(t) \) in Eq. (10a), (10b), and (10c) of their paper).

It is also important to notice that in their empirical application the authors introduce a nonnegativity boundary in the estimation of both real and nominal rates, which is inconvenient.
for real rates can well be negative.

Turning to our estimation procedure, pricing of coupons, principal and principal appreciation requires the estimation of the following 10 parameters: $\alpha, \tilde{\beta}, \sigma_i, \pi, \tilde{a}, \tilde{b}, \sigma_r, \pi_r, \rho$ and $k_t$. The parameters appear in Eq. (12), (13), and (15).

We have already justified the choice to set $k_t = 0$ in Subsection 3.4. In the next subsections, we describe the steps followed to estimate the remaining parameters. The "two-stage" procedure we adopt is described in De Felice and Moriconi (1991) for a model of the interest rate. We adjust their algorithm to our bivariate model.

### 4.1 Parameters for the Processes $i(t)$ and $r(t)$

The basis for inflation parameters estimation is the stochastic differential equation of $i(t)$ under the "real world" measure $P$,

$$di(t) = \alpha [\beta - i(t)] dt + \sigma_i dz_i^{P}(t).$$

This equation can be approximated by the following stochastic difference equation$^8$:

$$i_m = \beta_0^i + \beta_1^i i_{m-1} + \epsilon_{i,m}, \quad (21)$$

where

i. $i_m$ represents month $m$ estimate of $i(t)$,

ii. $\beta_0^i = \beta(1 - e^{-\alpha})$,

iii. $\beta_1^i = e^{-\alpha}$,

iv. $\epsilon_i$ is such that $\mathbb{E}[\epsilon_{i,m}] = 0$, $\mathbb{E}[\epsilon_{i,m}\epsilon_{i,m-1}] = 0$, and $\operatorname{Var}[\epsilon_{i,m}] = \Omega_i^2 = \sigma_i^2 \frac{1-e^{-2\alpha}}{2\alpha}$.

---

$^8$The approximation is feasible because the diffusion term, $\sigma_i$, is constant and the drift term, $\alpha[\beta - i(t)]$, is linear in $i(t)$. 
It follows immediately that \( \alpha = -(12 \ln \beta_1), \beta = \frac{\beta_0}{1-\beta_1}, \) and \( \sigma_i^2 = \frac{20\gamma^2(12 \ln \beta_1)}{(\beta_1)^2-1}. \) We can estimate \( \beta_0, \beta_1, \) and \( \Omega^2 \) through ordinary least squares to get asymptotically consistent estimates of \( \alpha, \beta, \) and \( \sigma_i. \) Values of \( i_m \) needed for regression are generated from the monthly series of CPI-AUC (see Subsection 5.1), through this simple transformation:

\[
i_m = \ln \left( \frac{\text{CPI-AUC}_m}{\text{CPI-AUC}_{m-12}} \right),
\]

i.e. \( i_m \) is the natural logarithm of the ratio between CPI-AUC of month \( m \) and CPI-AUC of month \( m - 12. \) These estimates of \( i(t) \) are expressed on an annual basis. The monthly series is calculated from January 1979 to December 2005.

The other state variable, \( r(t), \) presents a non constant diffusion term, \( \sigma_r \sqrt{r(t)}, \) in its stochastic differential; this feature does not immediately allow for a stochastic difference equation approximation as for \( i(t). \) The stochastic differential equation of \( r(t) \) under the "real world" measure \( P \) is

\[
dr(t) = a [b - r(t)] dt + \sigma_r \sqrt{r(t)} dz^P(t).
\]

Considering the new variable \( v(t) = \sqrt{r(t)}, \) and applying Ito formula, we find that \( dv(t) \) has a constant diffusion term, but its drift term is not linear in \( v(t). \) To linearize the drift, we approximate it with a first order Taylor polynomial whose initial point is the sample mean

\[
\overline{v(t)} = \frac{\sum_{t=1}^{N} \sqrt{r(t)}}{N} \text{ of } v(t).
\]

Finally, we get the following approximated "stochastic differential equation":

\[
dv(t) \approx [f(\overline{v(t)}) + f'(\overline{v(t)})(v(t) - \overline{v(t)})]dt + \frac{\sigma_r}{2} dz^P(t),
\]

where \( f(\cdot) \) indicates the drift of the exact stochastic differential of \( v(t). \) The approximating stochastic difference equation of the previous stochastic differential is

\[
\sqrt{r_m} = \beta_0^* + \beta_1^* \sqrt{r_{m-1}} + \varepsilon_{r,m}.
\]

Conditions similar to \( i., ii., iii., \) and \( iv. \) written after Eq. (21) hold. Making backwards transformations, it is possible to express parameters of Eq. (23) in terms of parameters of Eq. (22).
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(24); we have

\[ \sigma_r^2 = \frac{8 \Omega_r^2 (12 \ln \beta_1^r)}{(\beta_1^r)^2 - 1}, \]

\[ a = \left( -\frac{\beta_0^r (12 \ln \beta_1^r)}{v_m (\beta_1^r - 1)} - 2 (12 \ln \beta_1^r) \right), \]

and

\[ b = \frac{1}{a} \left( \frac{\beta_0^r (12 \ln \beta_1^r)}{\beta_1^r - 1} v_m + \frac{\sigma_r^2}{4} \right), \]

where \( \Omega_r^2 = \text{Var}[\varepsilon_{r,m}] \) and \( v_m = \frac{\sum_{t=1}^{N} \sqrt{r_{m-j+1}}}{N} \). Values of \( r_m \) needed for regression are generated from the monthly series of U.S. Department of the Treasury yields of actively traded non-inflation linked issues adjusted to have a constant maturity of 1 year (see Subsection 5.1). The monthly raw data, \( R_m \), are expressed on an annual basis and with discrete compounding; we get the continuously compounded estimate of \( r(t), r_m \), by this simple transformation

\[ r_m = \ln (1 + R_m). \] (25)

The monthly series of \( r_m \) is calculated from January 1979 to December 2005.

At this point, the time series of \( i_m \) and \( r_m \) are put together to jointly estimate \( \beta_0^i, \beta_1^i, \Omega_i^2, \beta_0^r, \beta_1^r, \Omega_r^2 \). The joint estimation method known as "seemingly unrelated regression" (SUR) is used (see Pindyck and Rubinfeld (1981)); this method applies to systems of regression equations with possibly correlated random errors but without dependent regressors, as is the case for our system:

\[ \begin{cases} 
    i_m = \beta_0^i + \beta_1^i i_{m-1} + \varepsilon_{i,m} \\
    \sqrt{r_m} = \beta_0^r + \beta_1^r \sqrt{r_{m-1}} + \varepsilon_{r,m} 
\end{cases} \] (26)

As a by-product of applying the SUR method, we are able to estimate the correlation between \( z^P_i(t) \) and \( z^P_r(t) \), \( \rho_{ir} \), as the month \( m \) estimated correlation between \( \varepsilon_i \) and \( \varepsilon_r \):

\[ \rho_{ir} = \rho_m(\varepsilon_i, \varepsilon_r). \]

Finally, to complete the picture, i.e. to get the values of \( \tilde{\beta}, \tilde{a}, \) and \( \tilde{b} \), we need to estimate two more parameters: interest and inflation risk premia, \( \lambda_i \) and \( \lambda_r \). The next subsection is devoted to lay down their estimation algorithms.
4.2 Interest and Inflation Risk Premia

The second step of our "two-stage" estimation algorithm concerns interest and inflation risk premia. We know that the Cox et al. (1985) model allows for a "closed form" expression of the "equivalent martingale measure" price of a zero coupon bond; this price can be represented by

\[ b(t, t + \tau) = \hat{\mathbb{E}}_t \left( e^{-\int_t^{t+\tau} \hat{r}(u) du} \right) = e^{-A(t, t + \tau)\hat{r}(t) + B(t, t + \tau)}, \]

where \( b(t, t + \tau) \) is the value at time \( t \) of a zero coupon bond expiring at \( t + \tau \), when it will pay 1 dollar (see Eq. (17), (18), and (19)).

If \( \hat{r}(t) \equiv \theta \), i.e. the process \( \hat{r}(t) \) is completely deterministic and, moreover, constant through time, we could express the value of the same zero coupon bond as \( b(t, t + \tau) = e^{-\theta \tau} \), with \( \theta \) representing the instantaneous nominal risk-free interest rate on, say, annual basis and \( \tau \) is time in years. Matching the two previous formulas, we get

\[ e^{-A(t, t + \tau)\hat{r}(t) + B(t, t + \tau)} = e^{-\theta \tau} \]

and, therefore,

\[ \theta(\hat{r}(t), t, t + \tau) = \frac{1}{\tau} [A(t, t + \tau)\hat{r}(t) - B(t, t + \tau)]. \]

The deterministic variable \( \theta \) is now referred to with the symbol \( \hat{\theta}(\hat{r}(t), t, t + \tau) \) to state that it is calculated on the basis of \( \hat{r}(t) \) and is related to the period \( t \) to \( t + \tau \). Expressions of \( A(t, t + \tau) \) and \( B(t, t + \tau) \) are based on the parameters estimated in the previous subsection, as well as on the interest risk premium we are going to estimate now.

We assumed in Section 3.1 a precise form for the interest risk premium: \( \lambda_r = \frac{\pi_r \sqrt{\hat{r}(t)}}{\sigma_r} \). De Felice and Moriconi (1991) suggest to solve the following minimization problem without constraints

\[ \min_{\pi_r} \sum_t \sum_{\tau} [\theta_0(t, t + \tau) - \theta(\hat{r}(t), t, t + \tau)]^2, \tag{27} \]

with respect to \( \pi_r \), which represents the unknown constant component of \( \lambda_r \); moreover:
t indicates the time at which θ₀ and θ are calculated;

τ indicates the residual life of the zero coupon bonds;

θ₀(t, t + τ) is the instantaneous yield observed on the market for a treasury bond with a constant maturity of τ years;

θ(φ(t), t, t + τ) is the instantaneous yield of the same treasury bond estimated on the basis of the Cox et al. (1985) model.

Each time we have solved problem (27) only one maturity of τ > 1 years has been chosen: the maturity of the treasury bond which better matched the expiration of the TIPS used to estimate the inflation risk premium.

The whole parameters estimation was executed without using TIPS’s market prices so far. However, these prices are now considered to "calibrate" inflation risk premium. The last parameter to be estimated, λᵢ = πᵢ, with πᵢ ∈ ℝ, is therefore treated as a degree of freedom of the whole "two-stage" algorithm; indeed, we leave inflation risk premium free to assume the value by which it makes the model price equal to the corresponding observed price of TIPS’s.

Suppose at time t the observed price of a TIPS is TIPS_{observed}^{t}; furthermore, suppose the model price of this TIPS is TIPS_{model}^{t} and is calculated through Eq. (11), after the change of measure from P to Q. We determine πᵢ* as the (numerical) solution of the following equation:

\[ \text{TIPS}_{i}^{\text{market}} - \text{TIPS}_{i}^{\text{model}}(\pi_i^*) = 0. \]  

(28)

Solutions πᵢ* and πᵢ* of problem (27) and Eq. (28) can be both positive and negative; therefore, also interest and inflation risk premia, λ, and λᵢ, can be of any sign; positive and negative risk premia are explained, for example, by the theory of "preferred habitat" (see Modigliani and Sutch (1966) and Modigliani and Sutch (1967)).
5 Empirical Validation

In this section, we propose some first empirical results of our model, both in terms of inflation risk premium estimation and in terms of pricing performance. A description of data on which the model is tested is put before.

5.1 Data Description

To test our model, we have collected the quotations of 4 TIPS’s for a total of 28 quarters over 7 years: each security has therefore been observed in 7 quarters over the period 1999-2005. The 4 TIPS’s expire in 2028, 2008, 2029, and 2009 and have been observed respectively in the first, second, third, and fourth quarter of every year. Table 1 summarizes the main technical features of our securities.

Insert Table 1 here

Estimation of model parameters is based on time series of consumer price index and yields of treasury securities. Table 2 summarizes technical features of the consumer price index used for TIPS’s indexation.

Insert Table 2 here

Estimation of American nominal interest parameters is done on monthly series of U.S. Department of the Treasury yields of actively traded non inflation linked issues adjusted to constant maturities of 1, 3, 5, 7, 10, and 30 years. These series cover the months from January 1979 to December 2005, according to data for CPI-AUC. Data are provided by the Federal Reserve Board. 1-year constant maturity bond yields are used for parameters estimation of Eq. (24), while the other constant maturity series are introduced for the minimization problem (27). Parameters estimates of models in Eq. (12) and (13) have been updated monthly, to coincide
with the release of inflation data, and starting every time from January 1979\(^9\). The U.S. Department of the Treasury issues bulletins of indexation coefficients halfway through every month. We remark that model formulas require, among other inputs, the time \( t \) estimates of the instantaneous interest and inflation rates. To proxy the former rate we have used daily data of government interest rates with a constant maturity of 1 year. For the latter estimates have been calculated daily as the (numerical) solution to the following equation, which compares theoretical and observed TIPS’s prices:

\[
TIPS_{t-1}^{\text{observed}} - TIPS_{t-1}^{\text{model}}(i_{t-1}^*) = 0.
\]

Solution \( i_{t-1}^* \) can be considered as a time \( t-1 \) market implied inflation rate, which we use as the best forecast for \( i_t \), i.e. the time \( t \) estimate of the instantaneous inflation rate.

### 5.2 Results

Market implied inflation rates are matched with historical inflation rates in Fig. 1. The former series is daily, while the latter is monthly. The two graphs run close to each other with the exception of 2003, when implied rates have been more volatile. It should be noted that CPI-AUC inflation rates summarize an historical view of the U.S. economy, while model implied inflation rates reveal a future market view.

Fig. 2 compares implied inflation, nominal and real interest rates. Real interest rates are obtained as a simple difference between nominal and inflation rates. When inflation runs faster than nominal rates, real interest rates become negative. This event does not appear to be rare in the U.S.. Indeed real interest rates were persistently negative in 2003 and 2004, and several times in 2001, 2002, and 2005.

\(^9\)The sample mean of month \( m \), \( \bar{\omega}_m = \frac{\sum_{j=1}^{N} \omega_{m-j+1}}{N} \), is needed for the estimation of parameters \( a \) and \( b \) of the nominal interest rate process. Its value has been calculated every month over a fixed window of \( N = 12 \) months, rather than from January 1979.
Our results on inflation rates are significantly different from those reported in Chen et al. (2005) especially in the period mid 2001 to mid 2003, where their inflation factor remains near zero, so that nominal and real interest rates strongly coincide. Such findings are quite puzzling and contrasting with official inflation data, most practitioners views (as reported in D’Amico, Kim and Wei (2007)) and monetary policy of the U.S. Federal Reserve System, which reacted to inflation rising risk by increasing nominal interest rates in early 2004.

**Insert Table 3 here**

Table 3 is also significant as it shows averages of the parameters estimates of models in (12) and (13). Averages are calculated over 16 annual estimations (4 TIPS’s × 4 quarters). In particular, the table shows the correlation between the driving Brownian motions of Eq. (2) and (3). As we can see, correlations are not always positive, as one would expect, but very low in absolute value. Negative numbers characterize constantly the interest risk premia; such finding implies a markup in the prices of TIPS’s. On the contrary inflation risk premia numbers are both negative and positive and in this last case they imply a discount on TIPS’s. During 2003 and 2004 the net effect of these premia generated a large discount on TIPS’s. If we look in more detail (see Fig. 3), we notice that positive inflation risk premia have been found mostly from late 2001 to 2005, that is about the same period when we also found negative real interest rates.

Finally, we comment on the pricing performance of our model. To match model prices with market prices, the latter have to be adjusted to account for accrued interest and inflation; the adjusted price is the "cum coupon" market price of a TIPS. According to market conventions, one has to sum the quoted "clean price" and the accrued interest from the last coupon detachment and the resulting sum is then multiplied by the inflation indexation coefficient. Fig. 4 to 7 show the pricing errors of our model; errors represent differences between model and market prices for the 4 TIPS’s we have investigated; we show only the quarters of years 1999,
2001, 2003, and 2005, the others being almost the same. Each quarter is referred to a particular
*TIPS*: the first quarter is related to the *TIPS* expiring in 2028, the second quarter shows model
performance for the *TIPS* maturing in 2008, the third quarter refers to the *TIPS* ending in 2029,
and the fourth quarter is for the *TIPS* due in 2009.

Our model allows for very small pricing errors, and Table 4 reports some summary statistics -
mean, standard deviation, maximum absolute value, explained percentage variance and root
mean squared error - confirming this performance. The displayed values are organized by
*TIPS*’s, and show better results for shorter maturity securities.

**Insert Table 4 here**

### 6 Conclusions

We propose a bivariate model for the pricing of inflation linked bonds. It specifies a Vasicek
dynamics for the inflation rate and a *CIR* process for nominal interest rates. It offers a closed
form equation for the price of inflation linked bonds with the exception of the embedded option
on the principal appreciation (which we calculate numerically). A distinct feature of our work is
that it can model economies during periods of negative inflation. Numerical calculations are used
to estimate inflation and interest risk premia. Consequently our model does not specify a
dynamics for real interest rates, nor it requires to estimate the corresponding term structure.

We discuss some conclusions with respect to the reference papers most close to our work, that is
Jarrow and Yildirim (2003) and Chen et al. (2005). As about the computational burden, the
comparison does not allow to determine a clear preference order. All models offer a closed form
solution for the price of *TIPS*’s. The paper of Jarrow and Yildirim (2003) contains a closed
formula also for inflation linked derivatives. However both Chen et al. (2005) and our approaches
offer an estimate of inflation risk premia which is not available in Jarrow and Yildirim (2003).

Both the Chen et al. (2005) and Jarrow and Yildirim (2003) works require a numerical
estimation of the term structure of real interest rates and a second estimation step for the model parameters. In our case we first estimate the model parameters and then apply a numerical procedure to identify the inflation risk premia.

The empirical analysis has offered several interesting results. Our model specification has been confirmed for the TIPS’s series analyzed here. Pricing performances are satisfactory for all the securities, especially with respect to TIPS’s with shorter maturities. Average pricing errors range from −0.00034 to 0.00893. These figures are significantly smaller compared to Chen et al. (2005). These authors adopt a single estimate run (based on the entire observation period) for all the parameters of their model. Such a choice is highly questionable (past theoretical prices are calculated through estimates based on future data) and can explain the lower pricing performances. Jarrow and Yildirim (2003) adopt daily parameter updating and obtain pricing precision comparable to ours, even though we adopt a monthly pace.

On the debate about real interest rates we have already discussed the theoretical opportunity of allowing negative inflation and real rates. Indeed we find empirical evidence of negative real interest rates contrasting with the estimation settings of Jarrow and Yildirim (2003), which impose a non negativity boundary. Our findings are also opposite to the theoretical settings in Chen et al. (2005), where both their inflation factor and the real rates are modeled as (strictly positive) CIR processes. Our evidence is not definitely conclusive on the superiority of one model on the other, since all the models cited here have been empirically validated. However the Chen et al. (2005) model produces the larger pricing errors exactly in the period (2003-2004) where our real interest rates estimates are mostly negative. In the case of Jarrow and Yildirim (2003) it would be interesting to extend their empirical analysis over year 2001 (which is the end of their sample period) to appreciate a full pricing performance comparison.
Appendix A - Tables

Table 1: Technical features of TIPS’s used in inflation risk premium estimation.

<table>
<thead>
<tr>
<th>Name</th>
<th>Id. Number (CUSIP)</th>
<th>Issue Date&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Maturity Date&lt;sup&gt;b&lt;/sup&gt;</th>
<th>Maturity</th>
<th>Coupon rate&lt;sup&gt;c&lt;/sup&gt;</th>
<th>IR&lt;sub&gt;Issue Date&lt;/sub&gt;&lt;sup&gt;d&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS_2008</td>
<td>912827317</td>
<td>01/15/98</td>
<td>01/15/08</td>
<td>10 years</td>
<td>3-5/8%</td>
<td>161.55484</td>
</tr>
<tr>
<td>TIPS_2028</td>
<td>912810FD5</td>
<td>04/15/98</td>
<td>04/15/28</td>
<td>30 years</td>
<td>3-5/8%</td>
<td>161.74000</td>
</tr>
<tr>
<td>TIPS_2009</td>
<td>9128274Y5</td>
<td>01/15/99</td>
<td>01/15/09</td>
<td>10 years</td>
<td>3-7/8%</td>
<td>164.00000</td>
</tr>
<tr>
<td>TIPS_2029</td>
<td>912810FH6</td>
<td>04/15/99</td>
<td>04/15/29</td>
<td>30 years</td>
<td>3-7/8%</td>
<td>164.39333</td>
</tr>
</tbody>
</table>

<sup>a</sup>Issue Date is denoted with g in the model (see Section 3).
<sup>b</sup>Maturity Date is denoted with H in the model (see Section 3).
<sup>c</sup>Coupons are paid semiannually.
<sup>d</sup>IR<sub>Issue Date</sub> is calculated according to Eq. (1).

Source of data: Datastream, by Thompson Financial, Inc.

Table 2: Technical features of Consumer Price Index - All Urban Consumers used for TIPS’s indexation.

<table>
<thead>
<tr>
<th>Name</th>
<th>Series Id.</th>
<th>Adjustment</th>
<th>Area</th>
<th>Item</th>
<th>Base Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumer Price Index</td>
<td>CUUR0000SA0</td>
<td>Not</td>
<td>U.S. city</td>
<td>All seasonally average items</td>
<td>1982-1984</td>
</tr>
<tr>
<td>- All Urban Consumers</td>
<td>(CPI-AUC)</td>
<td>adjusted</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Yearly averages and standard deviations of model parameters estimates.

<table>
<thead>
<tr>
<th>Year</th>
<th>$\rho_{ir}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma_{i}^2$</th>
<th>$\sigma_{r}^2$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\lambda_i$</th>
<th>$\lambda_r$</th>
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<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>-0.0299</td>
<td>0.0824</td>
<td>0.0057</td>
<td>0.0001</td>
<td>0.0036</td>
<td>0.1481</td>
<td>0.0644</td>
<td>-0.2605</td>
<td>-0.2209</td>
</tr>
<tr>
<td></td>
<td>(0.00345)</td>
<td>(0.00596)</td>
<td>(0.00488)</td>
<td>(0.0000009)</td>
<td>(0.00005)</td>
<td>(0.00269)</td>
<td>(0.00143)</td>
<td>(0.00504)</td>
<td>(0.00127)</td>
</tr>
<tr>
<td>2000</td>
<td>-0.0206</td>
<td>0.1013</td>
<td>0.0198</td>
<td>0.0001</td>
<td>0.0035</td>
<td>0.1702</td>
<td>0.0672</td>
<td>-0.3815</td>
<td>-0.37</td>
</tr>
<tr>
<td></td>
<td>(0.00356)</td>
<td>(0.00569)</td>
<td>(0.00367)</td>
<td>(0.0000012)</td>
<td>(0.00005)</td>
<td>(0.00705)</td>
<td>(0.00058)</td>
<td>(0.1949)</td>
<td>(0.16027)</td>
</tr>
<tr>
<td>2001</td>
<td>-0.0129</td>
<td>0.107</td>
<td>0.022</td>
<td>0.0001</td>
<td>0.0034</td>
<td>0.1396</td>
<td>0.0563</td>
<td>-0.332</td>
<td>-0.332</td>
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<tr>
<td></td>
<td>(0.00608)</td>
<td>(0.00151)</td>
<td>(0.00155)</td>
<td>(0.0000006)</td>
<td>(0.00003)</td>
<td>(0.02946)</td>
<td>(0.00821)</td>
<td>(0.17965)</td>
<td>(0.13535)</td>
</tr>
<tr>
<td>2002</td>
<td>-0.0062</td>
<td>0.0935</td>
<td>0.0113</td>
<td>0.0001</td>
<td>0.0034</td>
<td>0.0714</td>
<td>0.033</td>
<td>-0.251</td>
<td>-0.251</td>
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<tr>
<td></td>
<td>(0.00331)</td>
<td>(0.00513)</td>
<td>(0.00355)</td>
<td>(0.0000002)</td>
<td>(0.00003)</td>
<td>(0.00723)</td>
<td>(0.01042)</td>
<td>(0.20957)</td>
<td>(0.11168)</td>
</tr>
<tr>
<td>2003</td>
<td>-0.0047</td>
<td>0.1064</td>
<td>0.0185</td>
<td>0.0001</td>
<td>0.0032</td>
<td>0.0652</td>
<td>0.053</td>
<td>0.8895</td>
<td>-0.1852</td>
</tr>
<tr>
<td></td>
<td>(0.00354)</td>
<td>(0.00416)</td>
<td>(0.00268)</td>
<td>(0.0000005)</td>
<td>(0.00004)</td>
<td>(0.00278)</td>
<td>(0.00364)</td>
<td>(0.23643)</td>
<td>(0.01722)</td>
</tr>
<tr>
<td>2004</td>
<td>0.0033</td>
<td>0.1083</td>
<td>0.0187</td>
<td>0.0001</td>
<td>0.0032</td>
<td>0.0581</td>
<td>0.0244</td>
<td>0.429</td>
<td>-0.1734</td>
</tr>
<tr>
<td></td>
<td>(0.00331)</td>
<td>(0.00584)</td>
<td>(0.00323)</td>
<td>(0.0000009)</td>
<td>(0.00002)</td>
<td>(0.00724)</td>
<td>(0.01478)</td>
<td>(0.17409)</td>
<td>(0.01732)</td>
</tr>
<tr>
<td>2005</td>
<td>0.0049</td>
<td>0.1188</td>
<td>0.0239</td>
<td>0.0001</td>
<td>0.0032</td>
<td>0.074</td>
<td>0.0498</td>
<td>0.1875</td>
<td>-0.1982</td>
</tr>
<tr>
<td></td>
<td>(0.0039)</td>
<td>(0.00296)</td>
<td>(0.00201)</td>
<td>(0.0000008)</td>
<td>(0.00003)</td>
<td>(0.01176)</td>
<td>(0.00168)</td>
<td>(0.37873)</td>
<td>(0.02577)</td>
</tr>
</tbody>
</table>

Averages and (standard deviations) are calculated on sets of 16 parameters estimations performed each year: 4 estimations for every quarter. Parameters $\alpha$, $\beta$, $\sigma_{i}^2$, $\sigma_{r}^2$, and $\rho_{ir}$ are calculated through formulas presented in Subsection 4.1; inputs to these formulas are parameters $\beta_0^i$, $\beta_1^i$, $\Omega_1^i$, $\beta_0^r$, $\beta_1^r$, $\Omega_2^r$, and $\rho_m(\varepsilon_i, \varepsilon_r)$ of system (26). System (26) has been estimated by "seemingly unrelated regression" method.

Interest risk premia, $\lambda_r$, are estimated by solving problem (27); $\lambda_r = \pi_r \sqrt{\bar{r}(t)}/\sigma_r$

Inflation risk premia, $\lambda_i$, are estimated by solving problem (28); $\lambda_i = \pi_i$

Table 4: Summary statistics of pricing errors.

<table>
<thead>
<tr>
<th>Name</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Maximum$^a$</th>
<th>Explained percentage variance$^b$</th>
<th>RMSE$^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS_2008</td>
<td>-0.00649</td>
<td>0.19614</td>
<td>0.84465</td>
<td>0.99969</td>
<td>0.19602</td>
</tr>
<tr>
<td>TIPS_2009</td>
<td>0.00893</td>
<td>0.24211</td>
<td>1.57367</td>
<td>0.99953</td>
<td>0.24199</td>
</tr>
<tr>
<td>TIPS_2028</td>
<td>-0.04</td>
<td>0.55112</td>
<td>2.31525</td>
<td>0.99938</td>
<td>0.55193</td>
</tr>
<tr>
<td>TIPS_2029</td>
<td>-0.00044</td>
<td>0.71064</td>
<td>3.67554</td>
<td>0.99891</td>
<td>0.70983</td>
</tr>
<tr>
<td>Average</td>
<td>-0.0095</td>
<td>0.425</td>
<td>2.10228</td>
<td>0.99938</td>
<td>0.42494</td>
</tr>
</tbody>
</table>

Pricing errors are organized by security: for every TIPS, pricing errors of 7 quarters are put together for statistics calculations.

$^a$Column "Maximum" indicates the greatest absolute value pricing error.

$^b$Column "Explained percentage variance" is calculated as 1 minus the ratio of pricing errors variance to market prices variance.

$^c$RMSE is the square root of the average of squared pricing errors.
Appendix B - Figures

Figure 1: This figure shows model implied inflation rates and historical inflation rates. Data are drawn for the 7-year period 1999-2005. Both series collect instantaneously compounded rates: historical inflation rates are calculated by Eq. (22) every month and remain constant between two consecutive observations.
Figure 2: This figure shows model implied inflation rates as well as nominal and real interest rates. Nominal instantaneous interest rates are calculated by Eq. (25) from a daily series of yields of actively traded non-inflation linked issues adjusted to have a constant maturity of 1 year; real (instantaneous) interest rates represent the difference between nominal interest rates and inflation rates.
Figure 3: This figure shows the inflation risk premia estimated through Eq. (28) for all the TIPS's in the analyzed period (1999-2005). The shorter maturity TIPS's are TIPS_2008 and TIPS_2009, while the longer maturity TIPS's are the TIPS_2028 and the TIPS_2029. Each year 16 estimations have been performed. Inflation risk premia are mainly positive from 2001 to 2005.
Figure 4: TIPS’s pricing errors represent the difference between model and market prices; errors are expressed as percentage with respect to market prices. Every quarter is related to a different security. The 1st quarter collects prices about TIPS_2028, the 2nd quarter refers to TIPS_2008, the 3rd quarter shows prices of TIPS_2029, and the 4th quarter represents market and model prices of TIPS_2009.
Figure 5: TIPS's pricing errors represent the difference between model and market prices; errors are expressed as percentage with respect to market prices. Every quarter is related to a different security. The 1st quarter collects prices about TIPS_2028, the 2nd quarter refers to TIPS_2008, the 3rd quarter shows prices of TIPS_2029, and the 4th quarter represents market and model prices of TIPS_2009.
Figure 6: TIPS’s pricing errors represent the difference between model and market prices; errors are expressed as percentage with respect to market prices. Every quarter is related to a different security. The 1st quarter collects prices about TIPS_2028, the 2nd quarter refers to TIPS_2008, the 3rd quarter shows prices of TIPS_2029, and the 4th quarter represents market and model prices of TIPS_2009.
Figure 7: TIPS’s pricing errors represent the difference between model and market prices; errors are expressed as percentage with respect to market prices. Every quarter is related to a different security. The 1st quarter collects prices about TIPS_2028, the 2nd quarter refers to TIPS_2008, the 3rd quarter shows prices of TIPS_2009, and the 4th quarter represents market and model prices of TIPS_2009.
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