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A LÉVY PROCESS FOR THE GNIG PROBABILITY LAW WITH 2ND ORDER STOCHASTIC VOLATILITY AND APPLICATIONS TO OPTION PRICING

ANDERS ERIKSSON[†]

ABSTRACT. Here we derive the Lévy characteristic triplet for the GNIG probability law. This characterizes the corresponding Lévy process. In addition we derive equivalent martingale measures with which to price simple put and call options. This is done under two different equivalent martingale measures. We also present a multivariate Lévy process where the marginal probability distribution follows a GNIG Lévy process. The main contribution is, however, a stochastic process which is characterized by autocorrelation in moments equal and higher than two, here a multivariate specification is provided as well. The main tool for achieving this is to add an integrated Feller square root process to the dynamics of the second moment in a time-deformed Browninan motion. Applications to option pricing are also considered, and a brief discussion is held on the topic of estimation of the suggested process.

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1. Introduction

Since the outset the research agenda for financial modeling has been characterized by the usage of the Gaussian probability measure as a first probabilistic building block. One example from the field of option pricing is the celebrated article by Black and Scholes (1973), which can be claimed to have started the area of derivative pricing. An example from financial time series analysis is the autoregressive heteroscedastic process, or ARCH by Engle (1982), which was generalized later in Bollerslev (1986). This process is the fundament of the research agenda dealing with models with time varying higher moments. Both of these basic but path breaking models were subsequently improved so that the stochastic properties more closely resemble the observed financial time series; that is the assumption regarding Gaussianity is relaxed. For instance, option pricing models with empirically more valid processes can be found in Madan and Seneta (1990), Heston (1993) and Carr, Geman, Madan, and Yor (2004). As far as the literature on financial time series is concerned, examples are provided by Bollerslev (1987), Andersson (2001) and Eriksson (2005). The features common to all these extensions are that they allow for either excess kurtosis or skewness, or both.

In this paper we present an option pricing model based on a probability measure that can be interpreted as an extension of the normal inverse Gaussian probability measure, in particular it enables us to gain some flexibility in the probability measure by adding another scale parameter. We choose to call this the generalized normal inverse Gaussian probability measure (GNIG) to avoid any confusion. This probability measure corresponds to a stochastic jump process of the Lévy type. This is not the final port of call for this paper since there are overwhelming evidence that a Lévy process is only a partial solution to the problem of finding a stochastic process that mimics the behavior of the financial market, since, by definition it lacks any autocorrelation. Therefore we introduce a stochastic process where autocorrelation is allowed in the second moments (and higher). The process is defined along the lines that the so called theory of bi-power variation assumes, Barndorff-Nielsen and Shepard (2004c), which means that, like these authors, we assume a process for the log-price that is a sum of a jump process and a continuous stochastic volatility process. However, while Barndorff-Nielsen and Shepard (2004c) for bi-power variation theory assume a jump process with large and rare jumps for technical reasons and our suggested jump process assume an infinite number of jumps in a finite interval for further discussion, see Section 7. Altogether, the main contribution to the research agenda where bi-power variation is concerned is that it reveals a method with which to separate the continuous and the jump part of quadratic variation.

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In this context, an important question to ask is: Has the exact formulation of the stochastic process any influence on the option prices, or is this purely a theoretical exercise? The answer has not yet been answered fully, but some recent empirical evidence suggests that the answer is: Sometimes. In Schoutens, Simons, and Tistaert (2003) the authors calibrate a wide variety of option pricing models, mostly of the Lévy process with stochastic volatility type as presented in Carr, Geman, Madan, and Yor (2004). The pricing differences with respect to ordinary vanilla options are negligible. However, the differences when pricing exotic path dependent options is huge, which indicates that in such cases the specification of the stochastic process is of great importance in such cases. This suggests that when, the payoff function gets more complicated, the importance of the exact specification of the price process becomes more important.

This article could be interpreted as an attempt to include results from various areas of financial modeling. In particular, we consider the findings from financial time series analysis regarding the autocorrelation pattern for financial returns when we state the stochastic process for which we intend to price options. That is, we assume that autocorrelation is only relevant for higher moments (larger than or equal to two). The main contribution consists of the specification of Lévy processes with stochastic volatility without any autocorrelation spilling over into the mean dynamics. We also consider a multivariate version of this process. Applications to the area of option pricing are suggested. A minor contribution is the Lévy characterization of the GNIG probability measure.

The outline of the paper is as follows: In Section 2 we sets out the basic probabilistic preliminaries in the paper. In Section 3 the definitions and theorems relevant to the theory of Lévy processes are presented. In Section 4 we perform the Lévy characterization of the GNIG stochastic process. Section 5 is devoted to option pricing under the GNIG Lévy processes, where we suggest two different equivalent martingale measures. This section also contains some basic concepts of option pricing in general. Next, Section 6 contains a multivariate extension of the process suggested earlier and an application to option pricing is considered. In Section 7 we introduce stochastic volatility into the GNIG Lévy processes within the multivariate setting. The Fourier transforms are calculated for the processes obtained. A discussion regarding this type of stochastic volatility in Lévy processes is also provided. Section 8 contains concluding remarks and ideas concerning future work.

2. Probability spaces, filtration and stochastic processes

We need to make formal statements in the context of probability theory to characterize the process implied by the generalized normal inverse Gaussian probability law and to apply it to finance. We begin by defining the general probability space and then defining a stochastic process in this probability space. Variants of the definitions below can be found in books like Feller (1966), Karatzas and Shreve (1991), Billingsley (1995) and Protter (2004).

Definition 2.1 (General probability space). Suppose a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible outcomes and $\mathcal{F}$ is the sigma field associated with the probability space containing all relevant sets. $\mathbb{P}$ is the probability measure that generates the probability that such a relevant set in $\mathcal{F}$ will occur. Any $\mathcal{F}$ set A for which $\mathbb{P}(A)$ has the measure one is support for $\mathbb{P}$.

A sigma field is defined as a family of subsets of Ω closed under any countable collection of set operations. For a more detailed discussion about the construction of sigma fields, see Billingsley (1995) pp 30-32. In this paper we also assume that the probability space for our continuous time process is $\mathbb{P}$-complete. Because otherwise the characterization of the sample path becomes a problem, see e.g Billingsley (1995) pp 504-508. (For a definition of $\mathbb{P}$-completeness, see below.)

Definition 2.2 (Filtration). Define a general filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ associated with the above probability space, where $\mathbb{T} = \{0 \le t \le T : t \in [0, \infty)\}$, and where $\mathcal{F}_t$ is characterized by being an increasing sequence of sub sigma fields of $\mathcal{F}$.

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F} \text{ for } 0 \le s < t \le T$$

We assume that the following conditions to apply to the sigma field $\mathcal{F}$.

(1) $\mathcal{F}$ is complete (see definition below)
(2) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of Ω
(3) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ or, alternatively, $\mathbb{F}$ is right- continuous.

Definition 2.3 (Complete probability space). If, for each $B \subset A \in \mathcal{F}$ such that $P(A) = 0$ we have $B \in \mathcal{F}$, $\mathbb{P}$ is complete.

For more insight on complete probability spaces, see Billingsley (1995) pp 44-45.

Remark 2.1 ($\mathbb{P}$-completion). *A a procedure called $\mathbb{P}$-completion. exists. That is if we start with an incomplete probability space $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$, we can construct a complete probability space*

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$(\Omega, \mathcal{F}, \mathbb{P})$ *by setting* $\mathcal{F} = \sigma(\tilde{\mathcal{F}} \cup \mathfrak{Q})$

$$\mathfrak{Q} = \{B \subset \Omega : B \subset A \text{ for some } A \in \tilde{\mathcal{F}}, \text{ with } P(A) = 0\}$$

where $\sigma(\mathfrak{G})$ *denotes the smallest sigma field on* Ω *that contains* $\mathfrak{G}$

For greater insight into the above procedure, see Feller (1966) pp 123-124.

Definition 2.4 (Stochastic process)**.** Consider a stochastic process $Y = (Y_t)$ defined on the filtered probability space denoted by the following pentet $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, \mathbb{T})$. Recall that each $Y(t)$ is $\mathcal{F}$-adapted if $Y(t)$ is $\mathcal{F}_t$ measurable for each $t \in \mathbb{T}$. Further, we define the process Y as $\mathcal{F}$-predictable that is $Y(t)$ is $\mathcal{F}_{t-}$ measurable, which means that $Y(t)$ is known strictly before time t.

3. Basics of Lévy processes

We need to define infinitely divisible distributions to define a Lévy process. The reason for this is that, within this class of probability measures, we construct our Lévy process.

Definition 3.1 (Infinitely divisible distribution)**.** A probability distribution F is infinitely divisible if, for every n. a distribution F_n exists such that $F = F_n^{n*}$, where $*$ denotes the convolution of n F_n random variables.

Another way to express the concept of infinitely divisible distributions is by saying that F is infinitely divisible if and only if for each n the distribution can be represented as the distribution of the sum

$$\Psi_n = \phi_{1,n} + ... + \phi_{n,n}$$

of n independent random variables with a common distribution F_n. It is important to understand that the random variables, $\phi_{1,n}$ can be viewed as serving the purpose to simplify the notation and make things more intuitive. For a fixed n, $\phi_{1,n}, ..., \phi_{n,n}$ are assumed to be mutually independent, but the variables $\phi_{j,m}$ and $\phi_{k,n}$ with $m \neq n$ need not be defined in the same probability space. In other words, the joint probability measure does not need to exist.

Definition 3.2 (Lévy processes)**.** The adapted stochastic process

$$Y(t), \quad t \in [0, \infty], \quad Y(0) = 0$$

is a Lévy process if and only if

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(i): $Y(t)$ has increments which are independent of the past, that is $Y(t) - Y(s)$ is independent of $\mathcal{F}_s$ for $0 \le s < t < \infty$.

(ii): $Y(t)$ has stationary increment, that is $Y(t) - Y(s)$ has the same distribution as $Y(t-s)$.

One way of describing a Lévy process is to decompose it into two separate parts where one part is Brownian motion and the other is a mixture of compensated Poisson processes, see Theorems 40, 41 and 42 on pages 30 and 31 in Protter (2004). This leads the way to a formal characterization of the Lévy process using the Fourier transform: the celebrated Lévy-Khintchine formula.

Theorem 3.1 (Lévy-Khintchine formula). *Consider $\boldsymbol{\zeta} \in \mathbb{R}^d$, a positive semi-definite quadratic form Q on $\mathbb{R}^d$ and a measure $\boldsymbol{\Lambda}$ on $\mathbb{R}^d \backslash \{0\}$ such that $\int_{\mathbb{R}^d} \min(1, |y|^2) \boldsymbol{\Lambda}(dy) < \infty$. Further, for every $\boldsymbol{u} \in \mathbb{R}$ define $\kappa(\boldsymbol{u}) = \ln E[e^{-i\boldsymbol{u}\boldsymbol{Y(t)}}]$ where*

$$\kappa(\boldsymbol{u}) = i\langle \boldsymbol{\zeta}, \boldsymbol{u} \rangle + \frac{1}{2}Q(\boldsymbol{u}) - \int_{\mathbb{R}^d} (\exp(i\langle \boldsymbol{u}, \boldsymbol{x} \rangle) - 1 - i\langle \boldsymbol{u}, \boldsymbol{x} \rangle \mathbf{I}_{|y|<1}) \boldsymbol{\Lambda}(dy)$$

Then a unique probability measure $\mathbb{P}$ exists on Ω under which $\boldsymbol{Y}$ is a Lévy process and the jump process of $\boldsymbol{Y}$, $\Delta\boldsymbol{Y}$, is a Poisson point process with characteristic measure Λ. $< \cdot, \cdot >$ denotes the standard scalar product on a Euclidean space.

Proof: See Bertoin (1996) pp 13-15.

Remark 3.1 (Univariate Lévy-Khintchine formula). *Consider $\zeta \in \mathbb{R}$, $\nu \ge 0$ and Λ is a measure on $\mathbb{R} \backslash \{0\}$ such that $\int_{\mathbb{R}} (1 \wedge y^2) \Lambda(dy) < \infty$, $\kappa(u) = \ln E[e^{-iuY(t)}]$,*

$$\kappa(u) = i\zeta u + \frac{1}{2}\nu u^2 - \int_{\mathbb{R}} (\exp(iuy) - 1 - iuy \mathbf{I}_{|y|<1}) \Lambda(dy)$$

Then a unique probability measure $\mathbb{P}$ exists on Ω under which Y is a Lévy process. The jump process of Y, ΔY, is a Poisson point process with characteristic measure Λ.

From the above formula we can state what is called the Lévy characteristic triplet. That is, $[\zeta, \nu, \Lambda(dy)]$, where Λ is called the Lévy density of the process if the Lévy measure is of the form $\Lambda(dy) = \Lambda(y)dy$, i.e. if it is differentiable. The Lévy density has the same mathematical properties as a standard probability density except for the fact that it can be a divergent integral and must have no atom at zero. For more on the definition of the characteristic

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measure in a Lévy process, see Section B in Chapter 6.1 on Poisson random measures in Karatzas and Shreve (1991).

4. The Lévy characteristic triplet of the GNIG-law

According to Eriksson and Forsberg (2005), we can make an extension of the normal inverse Gaussian (NIG) distribution (see Barndorff-Nielsen (1978) and Barndorff-Nielsen (1997)) by adding a parameter that scales the variance in the derivation of the probability measure. The density function obtained is given in the following proposition.

Definition 4.1 (GNIG $(\lambda, \sigma, \delta, \gamma, \mu)$ probability law).

$$(4.1) \qquad f(y; \lambda, \sigma, \delta, \gamma, \mu) = \frac{\delta \sqrt{\frac{(\delta^2 + (y-\mu)^2)}{(\gamma^2 + \lambda^2)}} K_1\left(\sqrt{\frac{(\delta^2 + (y-\mu)^2)}{\sigma^2} \frac{(\gamma^2 + \lambda^2)}{\sigma^2}}\right) \exp\left(\delta\gamma + \frac{\lambda(y-\mu)}{\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}$$

where $y, \mu, \lambda, \sigma \in \mathbb{R}$ and $\delta, \gamma \in \mathbb{R}^+$. $K_1(.)$ denotes the modified Bessel function of third order and index one.

Remark 4.1 (The Fourier transform GNIG law). *The Fourier transform for the GNIG law is given by:*

$$(4.2) \qquad \varphi(s) = \exp(\delta(\gamma - (\gamma^2 + \sigma^2 s^2 - 2i\lambda s)^{\frac{1}{2}}) + \mu i s)$$

The GNIG law can be described as a normal mean-variance mixing law:

$$\mathcal{L}((\mu + \lambda V + \sigma \sqrt{V} Z)|V) = N(\lambda V + \mu, \sigma^2 V) \text{ where } \mathcal{L}(V) = IG(\delta, \gamma) \text{ and } \mathcal{L}(Z) = N(0, 1)$$

Denote an inverse Gaussian Lévy process by $V(t)$ with a cumulant generating function (CGF)[1]

$$(4.3) \qquad \kappa_{V(t)}(s) = t\delta(\gamma - \sqrt{(\gamma^2 - 2is)})$$

Now can we define the Lévy process corresponding to the GNIG probability law. This is done using classical subordination of Brownian motion.

Definition 4.2 (GNIG Lévy process).

$$Y(t) = \mu + \lambda V(t) + \sigma W(V(t))$$

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[1]Here we define the cumulant generating function (CGF) as the natural logarithm of the Fourier transform.

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where W(t)is an standard Brownian motion and $V(t)$ is an inverse Gaussian Lévy process, μ, λ and $\sigma \in \mathbb{R}$

The Fourier transform for the GNIG Lévy process with μ set to zero, can be expressed in the following way:

$$(4.4) \qquad \varphi_{Y(t)}(u) = \exp\{\kappa_{V(t)}((-i)f(u))\} \text{ where } f(u) = (\frac{1}{2}\sigma^2 u^2 - i\lambda u)$$

In order to define the process above, we derive the Lévy characteristic triplet. This is the standard way to characterize this kind of process.

Theorem 4.1 (Lévy characteristic triplet of the GNIG-law). *A Lévy characteristic triplet is said to be generated by a GNIG probability law if it is stated as:*

$$[\zeta, \nu, \Lambda]$$

where

$$(4.5) \qquad \zeta = \frac{\delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi} \int_0^1 \sinh(\frac{\lambda y}{\sigma^2}) K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})})) dy$$

$$(4.6) \qquad \nu = 0$$

$$(4.7) \qquad \Lambda = \frac{e^{\frac{\lambda y}{\sigma^2}} \delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi|y|} K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})})$$

Proof: See Appendix A

Compare the above result with the similar results for the normal inverse Gaussian probability law, where the Lévy measure, Λ_{NIG}, see Barndorff-Nielsen (1997) is:

$$\Lambda_{NIG} = \frac{\delta\gamma \exp(\beta y)}{\pi|y|} K_1(\gamma|y|)$$

According to the Lévy decomposition theorem (see Theorem 42 page 31 in Protter (2004)), the GNIG Levy-process can be expressed as:

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$$Y(t) = t\zeta + \int_{|y|<1} y p_t(dy) - t \int_{|y|<1} y \Lambda dy + \int_{|y|\geq1} y p_t(y) dy$$

$$= t\zeta + \underbrace{\int_{|y|<1} y p_t(dy) - E\{\int_{|y|<1} y p_t(dy)\}}_{Z^{(1)}} + \underbrace{\sum_{s<0\leq t} \Delta Y_s \mathbf{1}_{\{|\Delta Y_s|>1\}}}_{Z^{(2)}}$$

We define $p_t^{\{y<a\}} = \int_{\{y<a\}} p_t(dy)$ as a Poisson process with parameter $\Lambda(y)$ where $y < a$. $\Lambda(dy)$ is defined as the Lévy measure, see Remark 3.1. $Z^{(1)}$ can be interpreted as a jump martingale Lévy process consisting of a compensated Poisson process and $Z^{(2)}$ is a compounded Poisson process. The processes $Z^{(1)}$ and $Z^{(2)}$ are independent of each other since they are defined for different Borel sets: $\{\mathfrak{B}_1 : |y| \leq 1\}$ $\{\mathfrak{B}_2 : |y| > 1\}$, see p. 29 Theorem 39 of Protter (2004). This coincides with the well known general result that a Lévy process can be expressed as the sum of three independent Lévy processes, as explained on p. 15 in Bertoin (1996).

5. Option pricing under the GNIG Lévy process

In this section we derive two equivalent (or risk neutral) martingale measures that correspond to the GNIG Lévy process. The two measures in question are the so called Esscher measure, originally used in actuarial sciences, and the mean corrected martingale measure. These two measures can be regarded as standard tools for obtaining an equivalent martingale measure. We denote the physical measure as $\mathcal{P}$ and the corresponding risk neutral measure as $\mathcal{Q}$.

5.1. **Risk neutral valuation and market incompleteness.** We begin with a definition of an equivalent martingale measure.

Definition 5.1 (equivalent martingale measure)**.** A probability measure $\mathcal{Q}$ defined on $(\Omega, \mathbb{F})$ is an equivalent martingale measure if

- $\mathcal{Q}$ is equivalent to $\mathcal{P}$, that is, they have the same null sets.
- The discounted price process $\tilde{S}(t) = \exp(-rt)S(t)$ is a martingale under $\mathcal{Q}$.

To change our martingale measure the way suggested above implies some profound deep assumptions for the behavior of the agents on the market. In order to understand this central but abstract construction, it helps to observe the following: Risk aversion is equivalent to paying more attention to unpleasant states, that is, more unpleasant states are given an

increased probability of occurring. For example, people who are afraid of flying and therefore feel that there is a high probability that planes will crash are not irrational. They are just expressing their risk neutral probabilities. Thus, when we price options with a risk neutral measure, we can think of the agents on the market being risk neutral but with another set of probabilities than the one under the physical measure.

The question of existence of an equivalent martingale measure is strongly related to the absence of arbitrage on the market, while the issue of uniqueness has to do with whether the market is complete or not. One way of addressing the issue of market completeness is in the terms of the topology used to define the market (i.e. the space of cash flows) and the uniqueness of the state price densities. This can be a rather complicated and technical issue as is evident in Jarrow, Jin, and Madan (1999). However, we will use a less technical definition.

Definition 5.2 (Market completeness). A market is said to be complete if, for all integrable contingent claims, an admissible self-financing strategy that replicates the claim exists. Alternatively, the price of any derivative will be uniquely determined by the an absence of arbitrage requirement. In probabilistic terms, this means that, if the martingale measure has the predictable representation property (the measure is unique), then the market in question is complete.

For more on the predictable representation property of a martingale ,see pp 178-189 in Protter (2004). This property is delicate and exceptional. Examples of martingales with this property are Brownian motion and the compensated Poisson process. It is important to be aware that the uniqueness of the martingale measure implies the predictable representation property, which implies completeness. The opposite is not true because there are complete markets without any unique equivalent martingale measure.

5.1.1. *The Esscher measure.* One way of obtaining an equivalent martingale measure when the market is incomplete is to use the Esscher transform of the physical probability measure $\mathcal{P}$ to the risk neutral probability measure $\mathcal{Q}$. This particular measure is called the Esscher measure, and it is denoted $\mathcal{Q}^E$. The procedure obtain this measure is as follows.

Let Y be a random variable and $\varrho \in \mathbb{R} \setminus \{0\}$ where $E(e^{\varrho Y})$ exists. Construct a new positive random variable

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(5.1)
$$\widetilde{Y} = \frac{e^{\varrho Y}}{E(e^{\varrho Y})}$$

$\widetilde{Y}$ can be used as a Radon-Nikodym derivative which to define a new probability measure containing the same null sets as the old measure. Thus one obtains two measures that are equivalent. Define a measurable function ϵ. The expectation with respect to the new measure for the measurable function of the random variable Y, $\epsilon(Y)$ is:

(5.2)
$$E_{\widetilde{Y}}(\epsilon(Y); \varrho) = E_Y(\epsilon(Y)\widetilde{Y})$$

We derive a risk neutral probability measure with the above results, together with Radon-Nikodym Theorem, see Royden (1968) pp 276. Let $p_t(y)$ denote the probability density function under the physical measure. Then, for some ϱ (defined above), and using $\tilde{Y}$ as Radon Nikodym derivative, we can define a new probability law as follows:

(5.3)
$$\tilde{p}_t(y; \varrho) = \frac{\exp(\varrho y)p_t(y)}{\int_{\mathbb{R}} \exp(\varrho x)p_t(x)dx}$$

The above result can be used to derive a risk neutral probability measure for the GNIG law. We start out by assuming that we have a continuous dividend yield q and a continuously compounded short interest rate r. The parameter ϱ has to be chosen so that the discounted price process $S(t) = S(0)\exp(-(r-q)t + Y(t))$ is a martingale, i.e.,

$$S(0) = \exp(-(r-q)t)E_{\widetilde{S}(t)}(S(t); \varrho).$$

From this relation it can be shown (see Section 5 in Gerber and Shiu (1996)) that, in order for the martingale property of the measure to be fulfilled, the following relation must hold

(5.4)
$$\exp(r - q) = \frac{\varphi(-i(\varrho + 1))}{\varphi(-i\varrho)}$$

where φ denotes the Fourier transform of the $\mathcal{P}$ martingale measure. The solution to 5.4 is denoted ϱ^*. This parameter is used to define the equivalent martingale measure $\mathcal{Q}$.

Assume that the price process is defined as the exponential of the process $Y(t)$, i.e., $S(t) = \exp(Y(t))$, and that $E(\exp(\varrho Y(t))) = \varphi(\varrho)^t$ and $\exp(\varrho Y(t))/E(\exp(\varrho Y(t))) = S(t)^{\varrho}/E(S(t)^{\varrho})$. This makes it possible to state the following lemma from Gerber and Shiu (1996):

Lemma 5.1 (Factorization formula). *The expected value of the product of the stock price process raised to the power k, $S(t)^k$, and a measurable function $\epsilon(S(t))$ can be expressed as:*

$$E(S(t)^k \epsilon(S(t)); \varrho) = E(S(t)^k; \varrho) E(\epsilon(S(t)); \varrho + k)$$

Proof: See Gerber and Shiu (1996), p 188.

Continue by using the Esscher measure to derive an equivalent martingale measure for the GNIG process.

Proposition 5.1 ($\mathcal{Q}^E$ measure for the GNIG Lévy process). *The density function, $f^{\mathcal{Q}^E}(y)$, of the equivalent martingale measure $\mathcal{Q}^E$ (when t=1) can be expressed as:*

$$f^{\mathcal{Q}^E}(y) = \frac{\delta \sqrt{\frac{(\delta^2 + (y-\mu)^2)}{(\gamma^2 + \lambda^2)}} K_1(\sqrt{\frac{(\delta^2 + (y-\mu)^2)}{\sigma^2} \frac{(\gamma^2 + \lambda^2)}{\sigma^2}}) \exp(h(y; \delta, \gamma, \sigma, \lambda, \varrho^*))}{\sqrt{2\pi}\sigma}$$

where $y, \mu, \lambda \in \mathbb{R}$ and $\delta, \gamma \, \sigma \in \mathbb{R}^+$. $K_1(.)$ denotes the modified Bessel function of third order and index one.
Moreover $h(y; \delta, \gamma, \sigma, \lambda, \varrho^*) = \delta(\gamma^2 - \sigma^2 \varrho^{*2} - 2\lambda\varrho^*)^{\frac{1}{2}} + y(\varrho^* + \frac{\lambda}{\sigma^2}) - \mu(\varrho^* + \frac{\lambda}{\sigma^2})$ and ϱ^* is defined to be the solution with respect to ϱ of the following equation:

$$(r - q) = \delta([\gamma^2 - (\sigma\varrho)^2 - 2\lambda\varrho]^{\frac{1}{2}} - [\gamma^2 - (\sigma(\varrho+1))^2 - 2\lambda(\varrho+1)]^{\frac{1}{2}})$$

Hence ϱ^* is a function of $\mu, \lambda, \delta, \gamma \, \sigma, r$ and q

Proof: See Appendix B

In an attempt to gain an insight the characteristics of the $\mathcal{Q}^E$ measure and to make the difference between the $\mathcal{P}$ and $\mathcal{Q}^E$ measures more clear, we provide the expression for the cumulant generating function for the $\mathcal{Q}^E$ measure below.

Remark 5.1 (CGF for the $\mathcal{Q}^E$ measure). *The log of the Fourier transform (CGF) for the $\mathcal{Q}^E$ measure described in Proposition 5.1 can be expressed as:*

$$\kappa^{\mathcal{Q}^E}(s; \delta, \gamma, \sigma, \lambda, \mu, \varrho^*) = \chi + \delta(\gamma - (\gamma^2 - \sigma^2(is + \varrho^*)^2 - 2\lambda(is + \varrho^*))^{\frac{1}{2}}) + \mu(is + \varrho^*)$$

where χ is a constant that is not dependent on s.

Proof: See Appendix C

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5.1.2. *Mean correction of the exponentional of a Lévy process.* An alternative way of calculating an equivalent martingale measure is to use a location parameter, $\tilde{\mu}$. Denote this measure $\mathcal{Q}^{\tilde{\mu}}$. For the discounted exponential of the GNIG Lévy process a mean correction martingale measure is considered. This procedure will fulfill the conditions specified in Definition 5.1 so that we obtain a risk neutral martingale measure although this measure is different from the one obtained from the Esscher measure. Stochastic volatility Lévy processes are examples of occasions when the $\mathcal{Q}^{\tilde{\mu}}$ measure has been used to obtain an equivalent martingale measure, see Carr, Geman, Madan, and Yor (2004).

The effect that adding a location parameter has on the the Lévy characteristic triplet is to change the drift term. That is, $\tilde{\zeta} = \zeta + \tilde{\mu}$, while all other components remain unchanged. As in the case of the Esscher measure, continuous dividend yield q and a continuously compounded short rate r are assumed.

Proposition 5.2 ($\mathcal{Q}^{\tilde{\mu}}$ measure for the GNIG Lévy process)**.** *When the $\mathcal{P}$ measure is defined as a GNIG $(\lambda, \sigma, \delta, \gamma, \mu)$, then the $\mathcal{Q}^{\tilde{\mu}}$ measure is: (t=1)*

$$GNIG(\lambda, \sigma, \delta, \gamma, \tilde{\mu} + \mu)$$

where $\tilde{\mu} = -\mu + (r - q) - \delta(\gamma - (\gamma^2 - \sigma^2 - 2\lambda)^{\frac{1}{2}})$.

Proof: see Appendix D

5.2. **Option pricing under the $\mathcal{Q}^E$ and $\mathcal{Q}^{\tilde{\mu}}$ martingale measures.** To use the calculated equivalent martingale measures to price options, a theorem which establishes how we can use such a measure to price options is needed. This theorem is called the fundamental theorem of asset pricing, see for instance Delbaen and Schachermayer (1994). It is important to understand that the implied risk neutral stock price process is the exponential of the above discussed process for the log price (returns).

Theorem 5.1 (Fundamental theorem of asset pricing)**.**

$$\Upsilon(t) = E_{\mathcal{Q}}[\exp(-r(T - t)g(\{S(u)\ 0 \le u \le T\})|\mathcal{F}_t]$$

Denote the arbitrage-free price of the derivative at time $t \in [0, T]$ $\Upsilon(t)$. $\exp(-r(T - t))$ is called the discount factor. The expectation is calculated with respect to the equivalent martingale measure $\mathcal{Q}$. Further $\mathbb{F} = \{\mathcal{F}_t\ 0 \le t \le T\}$ is defined as the natural filtration of $S = \{S(t)\ 0 \le t \le T\}$. The function $g(\cdot)$ is called the payoff function and specifies which type of derivative that is priced.

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We clarify the application of the above theorem by using it to price a European call option. This is done via the derived density function for the $\mathcal{Q}^E$ equivalent martingale measure.

Example 5.1 (Pricing a European call option by means of the $\mathcal{Q}^E$ measure). To price a European call option with a strike price K at time $t = 0$ put $f = \max(S(T) - K, 0) = \max(S_0 \exp(Y(T) - K, 0)$. Using Lemma 5.1, the following is derived:

$$\Upsilon^C(T) = E_{\mathcal{Q}^\varepsilon}[\exp(-rT)\max(S(T) - K, 0)|\mathcal{F}_t]$$

$$= \exp(-rT)\int_0^\infty f_{\mathcal{Q}^E}(y, T; \varrho^*)\max(S(0)\exp(Y(T) - K)dy$$

$$(5.5) \qquad = \exp(-qT)S(0)\int_{\tilde{K}}^\infty f_{\mathcal{Q}^E}(y, T; \varrho^* + 1)dy - K\exp(-rT)\int_{\tilde{K}}^\infty f_{\mathcal{Q}^E}(y, T; \varrho^*))dy$$

where $\tilde{K} = \ln K - \ln S(0)$ and $\Upsilon^C(T)$ denotes the price of a call option with strike price K.

It is, of course, also possible to use the fundamental theorem of asset pricing to price a derivative under the $\mathcal{Q}^{\tilde{\mu}}$ measure.

6. A MULTIVARIATE STOCHASTIC PROCESS WITH GNIG LÉVY PROCESS MARGINALS

So far we have dealt with the univariate stochastic processes. Now we define a multivariate process. The basic idea is the same as when deriving a multivariate probability measure with a one-dimensional marginal for the GNIG law, see Eriksson and Forsberg (2005). In short, one assign a common subordination to each marginal. The random clock in the marginal Brownian motion consists of two parts: one part that is unique to the marginal in question and another part that is the same for all marginals. The sum then generates the actual subordination for the Brownian motion.

Definition 6.1 (Multivariate process). Define a multivariate process
$\mathbb{Y}(t) = [Y_1(t), ..., Y_k(t), ...Y_m(t)]$ where the $Y_k(t)$ process is defined as:

$$Y_k(t) = \{\mu_z + \mu_k\} + \{\omega_k V_z(t) + \tau_k W_z(V_z(t))\} + \{\omega_k V_k(t) + \tau_k W_k(V_k(t))\}$$

$$= \{\mu_z + \mu_k\} + \omega_k\{V_z(t) + V_k(t)\} + \tau_k\{W_z(V_z(t)) + W_k(V_k(t))\}$$

and $\mathcal{L}(V_z(t)) = IG(t\delta_z, \gamma)$, $\mathcal{L}(V_k(t)) = IG(t\delta_k, \gamma)$, $\mathcal{L}(W_{k,z}(t)) = N(0, t)$ and $\tau_k, \omega_k \in \mathbb{R}$. That is, $V_z(t)$ and $V_k(t)$ are inverse Gaussian Lévy processes. $W_z(t)$ and $W_k(t)$ are standard Brownian motions. All processes are independent.

Define the probability measure for the marginal process to show that it is of Lévy type. It will be in the GNIG class of probability measures.

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Proposition 6.1 (A probability measure for the marginal process). *The probability measure for the marginal process is*

$$\mathcal{L}(Y_k(t)) = GNIG(\omega_k, \tau_k, t(\delta_z + \delta_k), \gamma, \mu_z + \mu_k).$$

Hence the marginal process is a well defined Lévy process with the corresponding Lévy characteristic triplet when t=1.

$$[\zeta_k, \nu_k, \Lambda_k]$$

where

$$(6.1) \qquad \zeta_k = \frac{(\delta_z + \delta_k)\sqrt{\gamma^2 + \frac{\omega_k^2}{\tau_k^2}}}{\pi} \int_0^1 \sinh(\frac{\omega_k y_k}{\tau_k^2}) K_1(y_k \sqrt{(\frac{\gamma^2}{\tau_k^2} + \frac{\omega_k^2}{\tau_k^4})}) dy_k$$

$$(6.2) \qquad \nu_k = 0$$

$$(6.3) \qquad \Lambda_k = \frac{e^{\frac{\omega_k y}{\tau_k^2}}(\delta_z + \delta_k)\sqrt{\gamma^2 + \frac{\omega_k^2}{\tau_k^2}}}{|y_k|\pi} K_1(|y_k|\sqrt{(\frac{\gamma^2}{\tau_k^2} + \frac{\omega_k^2}{\tau_k^4})})$$

Proof: The proof follows directly from the definition of the GNIG Lévy process and from Theorem 4.1.

We will continue to characterize this process by deriving the multivariate Fourier transform of the process $\mathbb{Y}(t)$.

Proposition 6.2 (A multivariate Fourier transform). *If the location parameters μ_z and μ_k $\forall k$, are set to equal zero, then the Fourier transform for the multivariate stochastic process $\mathbb{Y}(t)$ is:*

$$\varphi_{\mathbb{Y}(t)}(\boldsymbol{s}) = \exp\{t[\delta_z(\gamma - (\gamma^2 - 2i\boldsymbol{s}'\boldsymbol{\omega} + (\boldsymbol{s}'\boldsymbol{\tau})^2)^{\frac{1}{2}}) + \sum_{k=1}^m \delta_k(\gamma - (\gamma^2 - 2is_k\omega_k + s_k^2\tau_k^2)^{\frac{1}{2}})]\}$$

where $\boldsymbol{s} = [s_1, ..., s_k, ..., s_m]', \boldsymbol{\tau} = [\tau_1, ..., \tau_k, ..., \tau_m]'$ and $\boldsymbol{\omega} = [\omega_1, ..., \omega_k, ..., \omega_m]'$

Proof: See Appendix E

Example 6.1 (Coefficient of correlation, bivariate case). Using the Fourier transform in Proposition 6.2 it can be shown that the coefficient of correlation for the bivariate case, here denoted $\rho_{[Y_1(t), Y_2(t)]}^{\mathbb{Y}(t)}$ has the following expression.

$$\rho_{[Y_1(t), Y_2(t)]}^{\mathbb{Y}(t)} = \frac{\delta_z(\omega_1\omega_2 + \tau_2\tau_1\gamma^2)}{\{(\omega_2^2 + \tau_2^2\gamma^2)(\delta_z + \delta_1)(\omega_1^2 + \tau_1^2\gamma^2)(\delta_z + \delta_2)\}^{\frac{1}{2}}}$$

One conclusion from Example 6.1 is that the sign of the correlation between two marginal processes is determined by the signs of the products $\tau_1\tau_2\gamma^2$ and $\omega_1\omega_2$ respectively. That is, if $|\omega_1\omega_2| \ll \tau_1\tau_2\gamma^2$, then in order to obtain a negatively correlated processes, τ_1 and τ_2 must have opposite signs.

Further, a remark addressing the issue of whether the proposed multivariate process is a Lévy process can be formulated.

Remark 6.1. *The process $\mathbb{Y}(t)$ with marginals corresponding to those in Proposition 6.1 is an m-dimensional Lévy process.*

Proof: See Appendix F

A trajectory for a bivariate GNIG Lévy process is illustrated below. With the marginal probability laws $\mathcal{L}(Y_1(t)) = GNIG(-\frac{1}{300}, \frac{37}{20}, t(1+\frac{2}{5}), 20, 0)$ and $\mathcal{L}(Y_2(t)) = GNIG(-\frac{1}{350}, \frac{17}{20}, t(1+\frac{1}{3}), 20, 0)$ and the $Corr[Y_1(t), Y_2(t)] = 0.732$.

[Insert figure 1 somewhere here]

In addition, a figure showing the trajectories for the inverse Gaussian subordinations that make up the bivariate process is illustrated below .

[Insert figure 2 somewhere here]

6.1. **Option pricing in a multivariate Lévy market.** It can often be problematic to try to squeeze an economic interpretation out of a probabilistic model. However, a brief financial meaning can be given to the probabilistic specification of this Lévy market. Assume that each marginal process is the process for a financial asset, that is, the log of a stock price. The price process then consists of two parts, a common factor that influences all assets in the Lévy market and one part that only has an impact on the asset in question. This partly coincides with the so called 'financial factor pricing models', see, for instance, Chapter 9 in Cochrane (2001) and Chapter 6 in Campbell, Lo, and MacKinlay (1997). The common subordination can be interpreted as being a common volatility factor applied to all assets in the market. That is, information that has an impact on all assets. The subordination that is unique to the marginal is information unique to the particular asset, in the literature on factor models this is often referred to as 'idiosyncratic' risk or noise, see, for example, page 72 in Campbell, Lo, and MacKinlay (1997).

The most straightforward way to obtain an equivalent martingale measure for a single asset in a Lévy market is to adopt the mean correction strategy. This gives an expression for the

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equivalent martingale measure for each marginal, yielding the risk neutral (under the $\mathcal{Q}^{\tilde{\mu}}$ measure) multivariate Lévy market.

Remark 6.2 ($\mathcal{Q}^{\tilde{\mu}}$ measure for the $\mathbb{Y}(t)$ process)**.** *If a multivariate process $\tilde{\mathbb{Y}}(t)$ has the following marginal process, then the process is said to be an equivalent martingale measure of the multivariate process, $\mathbb{Y}(t)$, from Definition 6.1.*

$$\mathcal{L}(\tilde{Y}_k(t)) = GNIG(\omega_k, \tau_k, t(\delta_z + \delta_k), \gamma, \tilde{\mu}_k + (\mu_k + \mu_z))$$

where $\tilde{\mu}_k = -(\mu_k + \mu_z) + (r - q) - (\delta_k + \delta_z)\{\gamma - (\gamma^2 - \tau_k^2 - 2\omega_k)^{\frac{1}{2}}\}$

Proof : The proof follows directly from Proposition 5.2

6.1.1. *Option pricing using the Fourier transform of the log stock price process.* The more complicated the process we assume for our stock-price process, the more unlikely it is to have an expression for the probability density function. However it is often the case that the Fourier transform for the density exists. The question of whether it is possible to price an option with the help the Fourier transform naturally arises, and the answer is that it is possible. The following theorem is attribute to Carr and Madan (1998):

Theorem 6.1 (Inversion of the modified call option price)**.**

$$\Upsilon_{K,T}^C = \frac{\exp(-\varepsilon)\ln K}{\pi} \int_0^{+\infty} \exp\left(-is\ln K h(s)\right)ds$$

where

$$h(s) = \frac{\exp(-rT)E[\exp(i(s - (\varepsilon + 1)i)Y(t))]}{\varepsilon^2 + \varepsilon - s^2 + i(2\varepsilon + 1)s}$$

and where $Y(t)$ is the log of the risk neutral stock price process. Further $\varepsilon \subseteq \mathbb{R}^+$ r denotes the short rate, T is the exercise time for the option and K is the strike price.

The parameter ε can be considered to be a damping coefficient in the transform of the modified call price, and needs to be determined in relation to the Fourier transform for the log stock price process, see page 69 in Carr and Madan (1998). The fast Fourier transform can be used to compute the option prices.

7. GNIG process with Stochastic Volatility

In this section when we use the term stochastic volatility process we mean an autoregressive stochastic volatility process. In a naive we may describe this as $\sigma_{t+1}^2 = \sigma_t^2 + \xi_t$, where $\{\xi_t\}$

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is an i.i.d. innovation sequence. Observe that there are always two ways of constructing such a process. You can either choose a probability measure for ξ_t, which then implies the distribution of σ_t^2 or the other way around, assume a probability measure for the σ_t^2 variable, implying the distribution for ξ_t. For more on the concept and definition of stochastic volatility, see the excellent article by Ghysels, Harvey, and Renault (1996).

Up to now we have been neglecting the issue of stochastic volatility when modeling financial data in an empirically valid manner. There is substantial evidence that financial data in general, and stock returns in particular, exhibit both jumps and stochastic volatility. Two very good empirical investigations into this matter can be found in Andersen, Benzoni, and Lund (2002) and Chernov and Ghysels (2000). These papers conclude that both jumps, for instance using a Lévy subordination and stochastic volatility (SV), for instance with a Feller square root process (see below). Further, there are some empirical findings that suggest that autocorrelation is something that has an impact on moments higher than or equal to two in the asset return probability law Eriksson (2005). At the very least the implied return process should not contain autocorrelation in the mean dynamics. This is to be interpreted as a conjecture. These things we will take into consideration when introducing SV in a Lévy process.

7.1. **A Lévy process with Stochastic Volatility.** To introduce SV into the Lévy process, the autocorrelated integrated Feller square root (IFSR) process is used. There are several advantages in using this process, for instance, both the Fourier transform of the process and the expression for the conditional probability measure are known. The integrated inverse Gaussian or the Gamma distribution Ornstein-Uhlenbeck processes would make excellent alternative candidates, see, for instance Barndorff-Nielsen, Nicolato, and Shephard (2002).

Definition 7.1 (Feller square root process)**.** The Feller square root process is the unique strong solution (see Karatzas and Shreve (1991) Chapter 5.2) of the following stochastic differential equation

$$(7.1) \qquad d\nu(t) = (\psi\nu(t) + \xi)dt + \varsigma\sqrt{\nu(t)}dW(t)$$

where $\psi \in \mathbb{R}$, $\xi, \varsigma \in \mathbb{R}^+$ and $W(t)$ is the standard Brownian motion.
Hence the integrated Feller square root process can be defined as:

$$(7.2) \qquad \zeta(t) = \int_0^t \nu(s)ds$$

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Remark 7.1 (Fourier transform IFSR process)**.**

$$(7.3) \qquad \varphi_{\zeta(t)}(s) = \frac{\exp\left(-\frac{\psi\xi}{\varsigma^2} + \frac{2is\nu_0}{g(s)\coth(\frac{tg(s)}{2})-\psi}\right)}{\{\cosh[\frac{t}{2}g(s)] - \frac{\psi}{g(s)}\sinh[\frac{t}{2}g(s)]\}^{\frac{2\xi}{\varsigma}}}$$

where $g(s) = \sqrt{\psi^2 - 2\varsigma^2 is}$

Proof: See Dufresne (2001) or Chapter 9 in Elliott and Kopp (1999).

This process has been used previously to obtain SV in continuous time processes. In the Heston model (see Heston (1993)) the main topic, was the pricing of options when the volatility of the asset followed the dynamics as described in Equation 7.1. In an excellent paper, Carr, Geman, Madan, and Yor (2004), stochastic volatility is introduced to a real valued Lévy process by making time in the Lévy process stochastic in accordance with Equation 7.2. However, one drawback of using this type of subordination approach is that the autocorrelated stochastic volatility process influences the expected value of the implied return process, which results in autocorrelation in the mean dynamics of the return process. Believing that autocorrelation in the returns process is only observed in higher moments, we should try to specify a process where stochastic volatility only has an impact on moments higher than or equal to two. This type of process we call 2nd order stochastic volatility, i.e. a process for the log price which implies a process for the log returns without any autocorrelation in the mean dynamics. An attempt to define such a process follows.

Definition 7.2 (GNIG Lévy process with 2nd order Stochastic Volatility)**.**

$$\tilde{Y}(t) = \lambda V(t) + W(\sigma^2 V(t) + \zeta(t))$$

where $W(t)$ is a standard Brownian motion and $V(t)$ is an inverse Gaussian Lévy process. $\zeta(t)$ is defined according to Equation 7.2 in definition 7.1

Proposition 7.1 (The Fourier transform of the $\tilde{Y}(t)$ process)**.**

$$\varphi_{\tilde{Y}(t)}(s) = \{\cosh[\frac{t}{2}\check{g}(s)] - \frac{\psi}{\check{g}(s)}\sinh[\frac{t}{2}\check{g}(s)]\}^{-\frac{2\xi}{\varsigma}}$$

$$\times \exp\{(\delta t\gamma - \frac{\psi\xi}{\varsigma^2}) + \frac{s^2\nu_0}{\check{g}(s)\coth(\frac{t\check{g}(s)}{2})-\psi} - \delta t[\gamma^2 - 2\lambda si + \sigma^2 s^2]^{\frac{1}{2}}\}$$

where $\check{g}(s) = \sqrt{\psi^2 - \varsigma^2 s^2}$

Proof: see Appendix G

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Option pricing requires that an we derive an equivalent martingale measure. Below is the Fourier transform of the $\mathcal{Q}^{\tilde{\mu}}$ measure presented.

Corollary 7.1 (The Fourier transform of the $\tilde{Y}(t)$ process under the $\mathcal{Q}^{\tilde{\mu}}$ measure).

$$\varphi_{\tilde{Y}(t),\mathcal{Q}^{\tilde{\mu}}}(s) = \{\cosh[\frac{t}{2}\breve{g}(s)] - \frac{\psi}{\breve{g}(s)}\sinh[\frac{t}{2}\breve{g}(s)]\}^{-\frac{2\xi}{\varsigma}}$$

$$\times \exp\{\frac{s^2\nu_0}{\breve{g}(s)\coth(\frac{t\breve{g}(s)}{2}) - \psi} - \delta t[\gamma^2 - 2\lambda si + \sigma^2 s^2]^{\frac{1}{2}} + is\tilde{\mu}\}$$

where $\breve{g}(s) = \sqrt{\psi^2 - \varsigma^2 s^2}$ and

$$\tilde{\mu} = (r-q) + \frac{2\xi}{\varsigma}\ln\{\cosh[\frac{t}{2}\sqrt{\psi^2 + 2\varsigma^2}] - \frac{\psi}{\sqrt{\psi^2 + 2\varsigma^2}}\sinh[\frac{t}{2}\sqrt{\psi^2 + 2\varsigma^2}]\} + \delta t[\gamma^2 - 2\lambda - \sigma^2]^{\frac{1}{2}}$$

$$+ \frac{\nu_0}{\sqrt{\psi^2 + 2\varsigma^2}\coth(\frac{t\sqrt{\psi^2 + 2\varsigma^2}}{2}) - \psi}$$

Proof. The proof follows directly from Proposition 7.1 and from the mean correction strategy given in Proposition 5.2. □

The above Fourier transform in conjunction with Theorem 6.1 can be used to price standard put and call options. Hence, a fairly straightforward formula has been obtained, which can be used to calibrate the risk neutral parameters.

Remark 7.2 (Implied univariate log return process).

$$Y(t) - Y(t-1) = R(t) = \lambda V(1) + W(\sigma^2 V(1) + [\zeta(t) - \zeta(t-1)])$$

Proof. The proof follows directly from Definition 7.2 □

We wish to determine the way in which this 2nd order stochastic volatility influences the sample path of the process we proceed in the following manner: First we simulate a log price process using a normal inverse Gaussian Lévy process with stochastic volatility as described in Carr, Geman, Madan, and Yor (2004). This is the same as making time in the NIG process follow an IFSR process. Secondly, the process described in Definition 7.2 is simulated. Then the implied time series for the log price differences is obtained and the empirical autocorrelation function for both of these series is calculated. This illustrates the first moment dynamics in the both processes.

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[Insert figure 3 somewhere here]

The parameter space for the process contains the risk neutral parameters corresponding to a NIG process with stochastic volatility (according to Carr, Geman, Madan, and Yor (2004)) for the Eurostoxx 50 index. These parameters are obtained by means of a calibration procedure; both parameters and procedure can be found in Schoutens, Simons, and Tistaert (2003). From the above figures, we see the distinct differences between the mean dynamics of the two process. For the NIG process with ordinary stochastic volatility, the autocorrelation is significant. In the case of 2nd order stochastic volatility, the mean dynamics are characterized by being a non-autocorrelated process. **Further, we provide a figure showing the trajectory for the 2nd order volatility process implied by the log return process. In order to compare this volatility process with a financial time series model, we provide the trajectory of the volatility process implied by a Student-t GARCH [2] Bollerslev (1987) in figure I. A brief comparison reveals that the two processes have similar dynamics in terms of the superficial behavior of the processes however an accurate comparison between these two models falls outside the scope of this paper**

[Insert figure 4 somewhere here]

[**Insert figure 5 somewhere here**]

7.2. **Lévy processes with Stochastic Volatility and the Leverage Effect. A desirable feature of a process set out to model the stochastic behavior of financial assets is that it is capable of capturing what is known as the leverage effect in stock returns. This effect is generally described as a negative correlation between return and future volatility innovations. Early empirical investigations of this includes Black (1976) and Christie (1982), these authors attribute this asymmetry stemming from changes in the debt to equity ratio also called financial leverage. Another explanation was put forward by French, Schwert, and Stambaugh (1987). An anticipated increase in volatility raises the required return on equity leading to an immediate stock price decline, this phenomena is sometimes referred to as volatility feedback [3]. As was the case of the debt to equity explanation this approach only provides a partial explanation to the**

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[2]the parameter space for this trajectory corresponds to a near IGARCH process

[3]The interpretation of volatility feedback is the following: if volatility is priced an anticipated increase in volatility raises the required return on equity leading to an immediate stock decline Wu (2001)

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leverage effect puzzle. In Campbell and Hentschel (1992) a combination of the two effects was suggested to explain asymmetry in stock market volatility however their explanation is also only partial. Bekaert and Wu (2000) conducted a comparison between the two effects and contributed the most of the asymmetry to the volatility feedback effect.

The process suggested here have a feature that resembles the volatility feedback effect, although it is the average or expected value of the unanticipated shocks to volatility multiplied with a real valued parameter (λ) that affect the mean of the returns. This also constitutes the source of skewness in the process.

In the above defined process feedback from volatility to the mean of the returns is determined by a function of the average shock to volatility, whether this can be viewed as volatility feedback in the context of the leverage effect is an open question. If you consider the process employed to obtain figure 3 and then you calculate the coefficient of correlation between the innovations of the implied log return and the square of the same sample path you obtain an estimate of the average impact of the volatility has to the mean dynamics. Using a simulated path of the size of 500000 observations the estimated correlation coefficient becomes approximately -0.175. This indicates that the feedback into the mean of the unanticipated shocks to volatility also yields a correlation structure between the first and second moment. However it is important to stress that this correlation structure is a function of the expected unanticipated shocks to volatility which differs from the classical definition of volatility feedback. A more elaborate study on the topic of leverage effect in the context of the suggested process is beyond the scope of this paper.

7.3. **A multivariate Lévy process with 2nd order Stochastic Volatility.** Let us now try to specify a multivariate version of the process from Definition 7.2. As before, we start by deciding how to define the marginal process. The procedure is quite simple: introduce an IFSR process into the variance of the common factor for the marginal process (compare with Definition 6.1).

Definition 7.3 (A multivariate Lévy process with Stochastic Volatility). Define a multivariate process

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$\tilde{\mathbb{Y}}(t) = [\tilde{Y}_1(t), ..., \tilde{Y}_k(t), ...\tilde{Y}_m(t)]'$, where the $\tilde{Y}_k(t)$ process is defined as:

$$\tilde{Y}_k(t) = \{\mu_z + \mu_k\} + \{\omega_k V_z(t) + \tau_k W_z(V_z(t) + \zeta(t))\} + \{\omega_k V_k(t) + \tau_k W_k(V_k(t))\}$$

and $\mathcal{L}(V_z(t)) = IG(t\delta_z, \gamma)$, $\mathcal{L}(V_k(t)) = IG(t\delta_k, \gamma)$, $\mathcal{L}(W_{k,z}(t)) = N(0, t)$ and $\tau_k, \omega_k \in \mathbb{R}$; that is, $V_z(t)$ and $V_k(t)$ are inverse Gaussian Lévy processes. $W_z(t)$ and $W_k(t)$ are standard Brownian motions, and $\zeta(t)$ is an IFSR process, see Definition 7.1.

Since any explicit expression of the probability measure of this process is unfeasible, we will provide the second best expression, that is the Fourier transform of the process in question.

Proposition 7.2 (A multivariate Fourier transform). *If the location parameters μ_z and μ_k $\forall k$ are set to zero, then the Fourier transform for the multivariate stochastic process $\tilde{\mathbb{Y}}(t)$ is:*

$$\varphi_{\tilde{\mathbb{Y}}(t)}(s) = \{\cosh[\frac{t}{2}g(s)] - \frac{\psi}{g(s)}\sinh[\frac{t}{2}g(s)]\}^{-\frac{2\xi}{\varsigma}}$$

$$\times \exp\{(\delta t\gamma - \frac{\psi\xi}{\varsigma^2}) + \frac{(s'\tau)^2\nu_0}{g(s)\coth(\frac{tg(s)}{2}) - \psi} - \delta t[\gamma^2 - 2is'\omega + (s'\tau)^2]^{\frac{1}{2}} + Q(s)\}$$

where $Q(s) = \sum_{k=1}^m \delta_k(\gamma - (\gamma^2 - 2is_k\omega_k + s_k^2\tau_k^2)^{\frac{1}{2}})$ and $g(s) = \sqrt{\psi^2 - \varsigma^2(s'\tau)^2}$ and $s = [s_1, ..., s_k, ..., s_m]'$, $\tau = [\tau_1, ..., \tau_k, ..., \tau_m]'$ and $\omega = [\omega_1, ..., \omega_k, ..., \omega_m]'$

Proof: See Appendix H.

You can, of course, use the same strategy as in the ordinary Lévy market to obtain an equivalent martingale measure and then price options with the aid of Proposition 6.1.

7.4. **Concerning the specification of the Lévy process with 2nd order Stochastic Volatility.** How should the stochastic process with 2nd order stochastic volatility be interpreted? The fact that we have a subordination of the Brownian motion that consists of the sum of a pure jump process, $V(t)$, and the SV process, $\zeta(t)$, coincides partly with the recent interesting theoretical findings of Barndorff-Nielsen and Shepard (2004c). They derived a asymptotic theory for a volatility process which is defined as jump part plus a stochastic volatility part. This yields a powerful tool for analyzing volatility in the financial market.

However Barndorff-Nielsen and Shepard (2004c) assume a finite number of jumps in a finite time interval in contrast to the general assumption regarding the jump process made in this paper. We assume that the jump process is an inverse Gaussian Lévy process which exhibits an infinite number of jumps in a finite time interval, that is, an infinite activity Lévy process.[4]

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[4]This property is confirmed determined by checking if the integral with respect to the Lévy measure is divergent or not.

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Although the authors conjecture that the results in their paper are valid for such an infinite activity process, they do not develop the arguments, instead, this conjecture is supported by the findings of Woerner (2003). These results are further developed in Barndorff-Nielsen and Shepard (2004a), Barndorff-Nielsen and Shepard (2004b) and Barndorff-Nielsen and Shepard (2004d). Altogether this opens up the possibility of the estimation of a Lévy process with 2nd order stochastic volatility using bi-power variation, for the moment it must be regarded as a conjecture.

When discussing the suggested process it is appropriate to raise the question of the economic motivation for the process. As mentioned earlier, when trying to interpret a probabilistic model in economic terms, we always face the risk of finding interpretations that are not really there. With this in mind, we can examine about how skewness enters the return process. The existence of asset returns with skewness is supported by, for instance, Simkowitz and Beedles (1980), Badrinith and Chatterjee (1988) and Peiró (1999).

The skewness that appear in this process is induced by the stochastic mean dynamics obtained by the inverse Gaussian Lévy process. This implies that the stochastic volatility process, here a Feller square root processes not directly contribute to the skewness. It reveals a belief that not the correlated volatility dynamics but only the unpredictable jumps from the inverse Gaussian Lévy process generate skewness in the return process.

If we assume that a high volatility period is more likely to have negative returns, then we obtain a probability measure where the coefficient for the jump process in the mean dynamics (in this setting denoted λ or ω) is negative. The jumps in volatility create the observed asymmetry or skewness. Another way to express this is that the jump process models the occurrence of 'unpredictable events' which not only have an impact on the volatility structure, but also have a direct negative impact on the mean dynamics of the return series. This results in a skewed process. The above mentioned coefficients can be said to measure the impact of the news arriving at the market. An investigation by Eriksson (2005) made in a discrete time setting where a GARCH (1,1) framework (see Bollerslev (1986)) is used to model the autocorrelation in the volatility, discovered a negative mean dynamic coefficient for the jump process. The parameter σ^2 enables the impact of the jump process to be different in the mean and in the second moment. Further, the impact of the jump process relative to the stochastic volatility process can also be monitored through this parameter.

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8. Concluding remarks and further work

In this paper we have presented a stochastic process that is a Lévy process with stochastic volatility, and which does not have any autocorrelation in the mean dynamics. This makes it possible to state an option pricing model with an underlying price process for which the implied log return process excludes autocorrelation for the mean dynamics. However, in this paper, there are two questions left unanswered. First there is the issue of estimation, and second, which empirical questions are relevant in the context of this kind of process. Should one concentrate on the calibration of the parameters under the $\mathcal{Q}$ process or is the estimation of the process under $\mathcal{P}$ more central?

The estimation of this kind of process is a complex issue. However, what has already been indicated in the previous section is that the paper by Barndorff-Nielsen and Shepard (2004c) has opened up the possibility of separating jumps and stochastic volatility under assumptions similar to the ones made in this paper. In particular, Barndorff-Nielsen and Shepard (2004c) conjecture that their results also hold for the type of process introduced in this paper. One interesting path of investigation is to see if their results can be used to estimate the process suggested in this paper. This should be regarded as future work.

Schoutens, Simons, and Tistaert (2003) raise an important empirical question. The authors obtain a perfect calibration of a volatility surface for standard call options for a wide range of complicated processes and they price these types of options fairly accurately, regardless of the process chosen. However, when they use more complicated payoff functions (so called exotic options), they get huge pricing errors which also vary with the assumption made about the process. This indicates that the details of the process assumptions are important for pricing exotic options, even though they are of minor importance for standard options. This raises the question: Can a minor adjustment to the autocorrelation in mean dynamics, such as we suggest in this paper, have a large influence on the pricing of exotic options? The areas of interest for future work identified in this paper naturally includes such an empirical investigation.

References

ANDERSEN, T. G., L. BENZONI, AND J. LUND (2002): "Am Empirical Investigation of Continuous-Time Equity Returns Models," *Journal of Finance*, 57, 1239–1284.

ANDERSSON, J. (2001): "On the Normal Inverse Gaussian Stochastic Volatility Model," *Journal of Business and Economic Statistics*, 19, 44–54.

BADRINITH, S., AND S. CHATTERJEE (1988): "On Measuring Skewness and Elongation in Common Stock Return Distributions: The case of the Market Index," *Journal of Business*, 61, 451–472.

BARNDORFF-NIELSEN, O. E. (1978): "Hyperbolic Distributions and Distributions on Hyperbolae," *Scandinavian Journal of Statistics*, 5, 151–157.

——— (1997): "Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling," *Scandinavian Journal of Statistics*, 24, 1–13.

BARNDORFF-NIELSEN, O. E., E. NICOLATO, AND N. SHEPHARD (2002): "Some Recent Developments in Stochastic Volatility Modeling," *Quantitative Finance*, 2, 11–23.

BARNDORFF-NIELSEN, O. E., AND N. SHEPARD (2004a): "Econometrics of testing for jumps in financial economics using bipower variation," *Unpublished manuscript, Nuffield College University of Oxford.*

——— (2004b): "Impacts of jumps on returns and realised variances: econometric analysis of time deformed Lèvy process," *Unpublished manuscript Nuffield College University of Oxford.*

——— (2004c): "Power and bipower variation with stochastic volatility and jumps," *Journal of Financial Econometrics*, 2, 1–48.

——— (2004d): "Power Variation and Time Change," *Unpublished manuscript Nuffield College University of Oxford.*

BEKAERT, G., AND G. WU (2000): "Asymmetric Volatility and Risk in Equity Markets," *Review of Financial Studies*, 13(1), 1–42.

BERTOIN, J. (1996): *Lévy Process.* Cambridge University Press, Cambridge.

BILLINGSLEY, P. (1995): *Probability and Measure.* John Wiley and sons, New York.

BLACK, F. (1976): "Studied in Stock Volatility Changes," *Proceedings of the 1976 meetings of The Business and Economics Statistics Section, American Statistical Association*, pp. 177–181.

BLACK, F., AND M. SCHOLES (1973): "The pricing of options and corporate liabilities," *Journal of Political Economy*, 81, 637–654.

BOLLERSLEV, T. (1986): "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 31, 307–327.

——— (1987): "A Conditionally Heteroscedastic Time Series Model for Speculative Prices and Rates of Return," *Review of Economics and Statistics*, 69(3), 542–546.

CAMPBELL, J., AND L. HENTSCHEL (1992): "No News is Good News: An Asymmetric model of Changing Volatility in Stock Returns," *Journal of Financial Financial Economics*, 31, 281–331.

CAMPBELL, J., A. W. LO, AND A. C. MACKINLAY (1997): *The Econometrics of Financial Markets.* Princeton University Press, Princeton.

CARR, P., H. GEMAN, D. B. MADAN, AND M. YOR (2004): "Stochastic Volatility for Lévy process," *Mathematical Finance*, 13, 345–382.

CARR, P., AND D. MADAN (1998): "Option valuation using the fast Fourier transform," *Computational Finance*, 2, 61–73.

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CHERNOV, M., AND E. GHYSELS (2000): "A Study towards a Unified Approach to the Joint Estimation
   of Objective and Risk Neutral Measures for the Purpose of Options Valuation," *Journal of Financial
   Economics*, 56, 407–458.

CHRISTIE, A. (1982): "The Stochastic Behavior of Common Stock Variances-Value, Leverage and Interest
   Rate Effects," *Journal of Financial Economics*, 10, 407–432.

COCHRANE, J. H. (2001): *Asset Pricing.* Princeton University Press, Princeton.

DELBAEN, F., AND W. SCHACHERMAYER (1994): "A general version of the fundamental theorem asset
   pricing," *Mathematische Annalen*, 300, 463–520.

DUFRESNE, D. (2001): "The Integrated Square-Root Process," *University of Montreal*, Research paper No.
   60.

ELLIOTT, R. J., AND P. E. KOPP (1999): *Mathematics of Financial Markets.* Springer- Verlag, New York.

ENGLE, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of
   United Kingdom Inflation," *Econometrica*, 50(4), 987–1007.

ERIKSSON, A. (2005): "The Mean Variance Mixing GARCH Model a new approach to quantify conditional
   skewness," *Unpublished manuscript, Uppsala University*.

ERIKSSON, A., AND L. FORSBERG (2005): "An Extension of the Generalized Hyperbolic Family of Proba-
   bility Laws," *Unpublished manuscript, Uppsala University*.

FELLER, W. (1966): *An introduction to Probability and Its Applications Vol 2.* Wiley, New York.

FRENCH, K., G. SCHWERT, AND R. STAMBAUGH (1987): "Expected Stock Returns and Volatility," *Journal
   of Financial Financial Economics*, 19, 3–30.

GERBER, H. U., AND E. S. SHIU (1996): "Acturial Bridges to dynamic hedging and option pricing,"
   Insurance: Mathematics and Economics, 18, 183–218.

GHYSELS, E., A. C. HARVEY, AND E. RENAULT (1996): "Stochastic Volatility," in *Handbook of Statistics*,
   ed. by C. RAO, , and G. Maddala, vol. 14, pp. 119–191, Amsterdam. North-Holland.

HALGREEN, C. (1979): "Self-Decomposability of the Generalized Inverse Gaussian and Hyperbolic Distri-
   butions," *Zeitschrift fur Wahrscheinlichkeitstheorie*, 47.

HESTON, S. L. (1993): "A Closed-Form Solution for Options with Stochastic Volatility with Application to
   Bond and Currency Options," *The Review of Financial Studies*, 6, 327–343.

JARROW, R. A., X. JIN, AND D. B. MADAN (1999): "The Second Fundamental Theorem of Asset Pricing,"
   Mathematical Finance, 9, 255–273.

KARATZAS, I., AND S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus.* Sprnger-Verlag, New
   York.

MADAN, D. B., AND E. SENETA (1990): "The Variance Gamma (V.G) Model for Share Market Returns,"
   The Journal of Business, 63, 511–524.

PEIRÓ, A. (1999): "Skewness in financial returns," *Journal of Banking and Finance*, 23, 847–862.

PROTTER, P. E. (2004): *Stochastic Integration and Differential Equations.* Springer- Verlag, Berlin.

ROYDEN, H. (1968): *Real Analysis.* MacMillan, New York.

SCHOUTENS, W., E. SIMONS, AND J. TISTAERT (2003): "A Perfect Calibration! Now What?," *UCS
   Technical Report K.U Leuven Leuven*, 2003-03.

SIMKOWITZ, M. A., AND W. L. BEEDLES (1980): "Assymetric Stable Distributed Security Returns,"
   Journal of American Statistical Association, 75, 306–312.

27

WOERNER, J. H. (2003): "Estimation of integrated volatility in stochastic volatility models," *Unpublished manuscript: OCIAM, Mathematical Institute, University of Oxford.*

WU, G. (2001): "The Determinants of Asymmetric Volatility," *The Review of Financial Studies*, 14, 837–859.

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APPENDIX A. PROOF OF THEOREM 4.1

We start off our proof by rewriting the log of the Fourier transform in the form of an integral. The reason for doing this is that the Lévy-Khintchine theorem is formulated as an integral. For the sake of simplicity we assume that μ in the GNIG law equals zero, for the same reason we also assume that t equals one.

Lemma A.1 (Integral representation of the CGF for the IG law).

$$\kappa(s) = \frac{\delta}{\pi\sqrt{2}} \int_{\gamma^2/2}^{\infty} \frac{1}{\sqrt{(y - \gamma^2/2)}} (-\ln(1 + \frac{is}{y})) dy$$

$$\text{(A.1)} \qquad = \frac{\delta}{\pi\sqrt{2}} \int_{\mathbb{R}^+} t^{-\frac{1}{2}} \{-\ln(1 + \frac{is}{t + \gamma^2/2})\} dt$$

This lemma is due to Halgreen (1979) page 15.

If we put $s = (-i)(\frac{1}{2}\tau^2 - i\lambda\tau)$ in A.1, we obtain as an integral representation of the CGF for the GNIG. This is given in the Remark A.1.

Remark A.1 (Integral representation of the CGF for the GNIG law).

$$\text{(A.2)} \qquad \ln\varphi(u) = \frac{\delta}{\pi\sqrt{2}} \int_{\mathbb{R}^+} t^{-\frac{1}{2}} \{-\ln(1 + \frac{\frac{1}{2}\sigma^2 u^2 - i\lambda u}{t + \gamma^2/2})\} dt$$

In order to continue we need to establish some way to rewrite Remark A.1 as an expression that resembles Remark 3.1. Therefore we state the Lemma A.2.

Lemma A.2.

$$\text{(A.3)} \qquad -\ln(1 + \frac{(\frac{1}{2}\sigma^2 u^2 - i\lambda u)}{\theta}) = \int_{\mathbb{R}} \frac{1}{|y|} \exp\{-(\frac{2\theta}{\sigma^2} + \frac{\lambda^2}{\sigma^4})^{\frac{1}{2}} |y|\} (\exp(iuy) - 1) e^{\frac{\lambda y}{\sigma^2}} dy$$

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Proof of Lemma A.2. Start by differentiating t0he left hand side of A.3

$$(A.4) \qquad -\frac{d\{-\ln(1+\frac{(\frac{1}{2}\sigma^2u^2-i\lambda u)}{\theta})\}}{du} = \frac{\frac{\sigma^2u-i\lambda}{\theta}}{1+\frac{\frac{1}{2}\sigma^2u^2-i\lambda u}{\theta}}$$

$$= \frac{\frac{2\theta}{\sigma^2}(\frac{\sigma^2u-i\lambda}{\theta})}{\frac{2\theta}{\sigma^2}(1+\frac{\frac{1}{2}\sigma^2u^2-i\lambda u}{\theta})}$$

$$= \frac{2(u-i\frac{\lambda}{\sigma^2})}{\frac{2\theta}{\sigma^2}+u^2-2ui\frac{\lambda}{\sigma^2}}$$

$$= \frac{2(u-i\frac{\lambda}{\sigma^2})}{(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})+(u-i\frac{\lambda}{\sigma^2})^2}$$

$$= 2\int_{\mathbb{R}^+}\exp\{-(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}y\}\{-\sin(y(u-i\frac{\lambda}{\sigma^2})\}dy$$

Integration of A.4 yields

$$-\ln(1+\frac{(\frac{1}{2}\sigma^2u^2-i\lambda u)}{\theta})$$

$$= 2\int_{\mathbb{R}^+}\exp\{-(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}y\}\int_0^u-\sin(y(q-i\frac{\lambda}{\sigma^2}))dqdy$$

$$= 2\int_{\mathbb{R}^+}\exp\{-(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}y\}\frac{\cos(y(u-i\frac{\lambda}{\sigma^2}))-\cosh(\frac{\lambda}{\sigma^2}y)}{y}dy$$

$$= \int_{\mathbb{R}^+}\frac{1}{y}\exp\{-(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}y\}\{(\exp(iuy)-1)e^{\frac{\lambda y}{\sigma^2}}+(\exp(-iuy)-1)e^{-\frac{\lambda y}{\sigma^2}}\}dy$$

$$= \int_{\mathbb{R}}\frac{1}{|y|}\exp\{-(\frac{2\theta}{\sigma^2}+\frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}|y|\}(\exp(iuy)-1)e^{\frac{\lambda y}{\sigma^2}}dy$$

$\square$

Let us now use (A.3), (A.2) and the Lévy Khintchine theorem to obtain a representation of a Lévy-process that will enable us to calculate the Lévy-triplet.

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Proof of Theorem 4.1.

$$\ln \varphi(u) = \frac{\delta}{\pi\sqrt{2}} \int_{\mathbb{R}^+} t^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{1}{|y|} \exp\{-(\frac{2t+\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})^{\frac{1}{2}}|y|\}(\exp(iuy)-1)e^{\frac{\lambda y}{\sigma^2}}dydt$$

$$= \frac{\sqrt{2}\delta}{\pi} \int_{\mathbb{R}} \frac{\exp(iuy)-1}{|y|} e^{\frac{\lambda y}{\sigma^2}} \int_{\mathbb{R}^+} \exp\{-\frac{|y|\sqrt{2}}{\sigma}(s^2 + \frac{1}{2}(\gamma^2 + \frac{\lambda^2}{\sigma^2}))^{\frac{1}{2}}\}dsdy$$

$$= \frac{\delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi} \int_{\mathbb{R}} (\exp(iuy)-1)|y|^{-1}e^{\frac{\lambda y}{\sigma^2}} K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})})dy$$

$$= \int_{\mathbb{R}} (\exp(iuy)-1) \overbrace{\frac{e^{\frac{\lambda y}{\sigma^2}}\delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi|y|} K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})})}^{\Lambda} dy$$

$$= \int_{|y|\geq 1} (\exp(iuy)-1)\Lambda dy + \int_{|y|<1} (\exp(iuy)-1-iuy)\Lambda dy + iu\zeta$$

where

$$\zeta = \frac{\delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi} \int_{|y|<1} \frac{y}{|y|}e^{\frac{\lambda y}{\sigma^2}} K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})})dy$$

$$= \frac{\delta\sqrt{\gamma^2 + \frac{\lambda^2}{\sigma^2}}}{\pi} \int_0^1 \sinh(\frac{\lambda y}{\sigma^2}) K_1(|y|\sqrt{(\frac{\gamma^2}{\sigma^2} + \frac{\lambda^2}{\sigma^4})}))dy$$

□

Appendix B. Proof of Proposition 5.1

Proof. We start out by deriving the density function for the $\mathcal{Q}^E$ probability measure. Using Remark 4.1 and the expression (5.3) we get

$$f^{\mathcal{Q}^E}(y) = \frac{\exp(\varrho^* y)f(y; \lambda, \sigma, \delta, \gamma, \mu)}{E(\exp(\varrho^* Y))}$$

$$= \frac{\exp(\varrho^* y)f(y; \lambda, \sigma, \delta, \gamma, \mu)}{\exp(\delta\gamma - \delta(\gamma^2 - \sigma^2\varrho^{*2} - 2\lambda\varrho^*)^{\frac{1}{2}} + \mu\varrho^*)}$$

$$= \frac{\exp(\varrho^* y)\frac{\delta\sqrt{\frac{(\delta^2+(y-\mu)^2)}{(\gamma^2+\lambda^2)}}K_1(\sqrt{\frac{(\delta^2+(y-\mu)^2)}{\sigma^2}\frac{(\gamma^2+\lambda^2)}{\sigma^2}})\exp(\delta\gamma+\frac{\lambda(y-\mu)}{\sigma^2})}{\sqrt{2\pi}\sigma}}{\exp(\delta\gamma - \delta(\gamma^2 - \sigma^2\varrho^{*2} - 2\lambda\varrho^*)^{\frac{1}{2}} + \mu\varrho^*)}$$

(B.1)
$$= \frac{\delta\sqrt{\frac{(\delta^2+(y-\mu)^2)}{(\gamma^2+\lambda^2)}}K_1(\sqrt{\frac{(\delta^2+(y-\mu)^2)}{\sigma^2}\frac{(\gamma^2+\lambda^2)}{\sigma^2}})\exp(h(y; \delta, \gamma, \sigma, \lambda, \varrho^*)))}{\sqrt{2\pi}\sigma}$$

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where $h(y; \delta, \gamma, \sigma, \lambda, \varrho^*) = \delta(\gamma^2 - \sigma^2\varrho^{*2} - 2\lambda\varrho^*)^{\frac{1}{2}} + y(\varrho^* + \frac{\lambda}{\sigma^2}) - \mu(\varrho^* + \frac{\lambda}{\sigma^2})$

Let us continue with the proof for the function related to the ϱ^* parameter. Start out by applying 5.4 using the Fourier transform from Remark 4.1.

$$\exp(r - q) = \frac{\exp\{\delta(\gamma - [\gamma^2 + \sigma^2(-i(\varrho + 1))^2 - 2i\lambda(-i(\varrho + 1))]^{\frac{1}{2}})\}}{\exp\{\delta(\gamma - [\gamma^2 + \sigma^2(-i\varrho)^2 - 2i\lambda(-i)\varrho]^{\frac{1}{2}})\}}$$

$$= \exp\{\delta([\gamma^2 - \sigma^2\varrho^2 - 2\lambda\varrho]^{\frac{1}{2}} - [\gamma^2 - \sigma^2(\varrho + 1)^2 - 2\lambda(\varrho + 1)]^{\frac{1}{2}})\}$$

$$\Leftrightarrow$$

(B.2) $$(r - q) = \delta([\gamma^2 - (\sigma\varrho)^2 - 2\lambda\varrho]^{\frac{1}{2}} - [\gamma^2 - (\sigma(\varrho + 1))^2 - 2\lambda(\varrho + 1)]^{\frac{1}{2}})$$

$\square$

Appendix C. Proof of Remark 5.1

Proof. The first step consists of calculating the Fourier transform of the expression in Proposition 5.1.

$$\varphi_{\mathcal{Q}^E}(s; .) = \int_{\mathbb{R}} f^{\mathcal{Q}^E}(y) \exp(isy)(dy)$$

$$= \int_{\mathbb{R}} \underbrace{\exp(\delta(\gamma^2 - \sigma^2\varrho^{*2} - 2\lambda\varrho^*)^{\frac{1}{2}} - \delta\gamma) + \mu\varrho^*)}_{\tilde{\chi}} f^{\mathcal{P}}(y) \exp((is + \varrho^*)y)(dy)$$

$$= \tilde{\chi}\varphi_{\mathcal{P}}(-i(is + \varrho^*))$$

(C.1) $$= \tilde{\chi} \exp(\delta(\gamma - (\gamma^2 - \sigma^2(is + \varrho^*)^2 - 2\lambda(is + \varrho^*))^{\frac{1}{2}}) + \mu(is + \varrho^*))$$

Then the expression for the CGF follows directly from: C.1

$$\kappa^{\mathcal{Q}^E}(s; \delta, \gamma, \sigma, \lambda, \mu, \varrho^*) = \chi + \delta(\gamma - (\gamma^2 - \sigma^2(is + \varrho^*)^2 - 2\lambda(is + \varrho^*))^{\frac{1}{2}}) + \mu(is + \varrho^*)$$

$\square$

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APPENDIX D. PROOF OF PROPOSITION 5.2

Proof. Denote the $GNIG(\lambda, \sigma, t\delta, \gamma, \mu)$ process at time $t = 1$ by $Y(1)$. Now define the expected value of the discounted stock process as:

$$\Psi = E[\exp\{Y(1) - (r - q)\}]$$

In order to change the stock price process in such a way that the price process is still a martingale, we define a new parameter $\tilde{\mu}$ such that:

(D.1) $$\Psi = e^{-\tilde{\mu}}$$

Solving for $\tilde{\mu}$ in D.1 yields:

$$\tilde{\mu} = -\ln \Psi = (r - q) - \ln(\varphi(-i)) = (r - q) - \mu - \delta(\gamma - (\gamma^2 - \sigma^2 - 2\lambda)^{\frac{1}{2}})$$

Now, we can use the derived parameter $\tilde{\mu}$ to change the price process $(e^{Y(1)})$ in such a manner that it continues to be a martingale. We obtain the following $\mathcal{Q}^{\tilde{\mu}}$ martingale measure.

$$\exp\{Y(1) + \tilde{\mu}\}$$

Hence the process for the log price under the $\mathcal{Q}^{\tilde{\mu}}$ measure is $GNIG(\lambda, \sigma, \delta, \gamma, \tilde{\mu} + \mu)$ □

APPENDIX E. PROOF OF PROPOSITION 6.2

Proof. Let us define $Q_k(t) = \lambda_k V_k(t) + \tau_k W_k(V_k(t))$ and $\mathbb{Q}'(t) = [Q_1(t), ..., Q_k(t), ..., Q_m(t)]$

$E[\exp\{i\boldsymbol{s}'\mathbb{Y}(t)\}] =$

$E[\exp\{i\boldsymbol{s}'[\omega_1 V_z(t) + \tau_1 W_z(V_z(t)) + Q_1(t)), \ ... \ , \omega_m V_z(t) + \tau_m W_z(V_z(t)) + Q_m(t)]'\} =$

$E[\exp\{i\boldsymbol{s}'\boldsymbol{\omega}V_z(t) + i\boldsymbol{s}'\boldsymbol{\tau}W_z(V_z(t))\} + i\boldsymbol{s}'\mathbb{Q}(t)]$

Let us now continue by defining the CGF for the $V_z(t)$ process as $\kappa_{V_z(t)}(u)$ and the corresponding function for the $V_k(t)$ process as $\kappa_{V_k(t)}(u)$. Both are defined according to Equation 4.3. Then the above expectation can be expressed as:

$$E[\exp(\boldsymbol{s}'\mathbb{Y})(t)] = \exp\{\kappa_{V_z(t)}(-i(i\boldsymbol{s}'\boldsymbol{\omega} + \frac{(i\boldsymbol{s}'\boldsymbol{\tau})^2}{2}) + \sum_{k=1}^{m} \kappa_{V_k(t)}(-i(is_k\omega_k - \frac{\tau_k^2 s_k^2}{2}))\}$$

$$= \exp\{t[\delta_z(\gamma - (\gamma^2 - 2i\boldsymbol{s}'\boldsymbol{\omega} + (\boldsymbol{s}'\boldsymbol{\tau})^2)^{\frac{1}{2}}) + \sum_{k=1}^{m} \delta_k(\gamma - (\gamma^2 - 2is_k\omega_k + s_k^2\tau_k^2)^{\frac{1}{2}})]\}$$

□

APPENDIX F. PROOF OF REMARK 6.1

Let us use the Multivariate Fourier transform from Proposition 6.2 to ensure that the criteria from Definition 3.2 is fulfilled.

Since the Fourier transform can be written in the form:

$$\varphi_{\mathbb{Y}(t)}(\boldsymbol{s}) = \exp\{tf(\boldsymbol{s};\boldsymbol{\theta})\}$$

where $\boldsymbol{\theta}$ denotes an m-dimensional parameter vector. We can conclude that criteria **(i)** in Definition 3.2 is fulfilled since :

$$Y(t)-Y(s) = Y(t)-Y(t-\xi) = Y(\xi) \Leftrightarrow \exp\{tf(\boldsymbol{s};\boldsymbol{\theta})\}\exp\{-(t-\xi)f(\boldsymbol{s};\boldsymbol{\theta})\} = \exp\{\xi tf(\boldsymbol{s};\boldsymbol{\theta})\}$$

Therefore $Y(t)-Y(s)$ is independent of the filtration $\mathcal{F}_s$, since it only depends on the length of the interval ξ. Criteria **(ii)** in Definition 3.2 follows directly from the above.

Hence the process $\mathbb{Y}(t)$ is a m-dimensional Lévy process.

APPENDIX G. PROOF OF PROPOSITION 7.1

Proof.

$$\begin{aligned}
\varphi_{\tilde{Y}(t)}(s) =& E[\exp\{is(\lambda V(t) + W(\sigma^2 V(t) + \zeta(t)))\}]\\
=& E_{V(t),\zeta(t}[\exp\{is\lambda V(t) + \frac{s^2\sigma^2 V(t)}{2} + \frac{\zeta(t)s^2}{2}\}]\\
=& \varphi_{Y(t)}(s)\varphi_{\zeta(t)}(-i\frac{s^2}{2})
\end{aligned}$$

where $\varphi_{Y(t)}(s)$ is given in Remark 4.1 and $\varphi_{\zeta(t)}(s)$ is given in Remark 7.1 $\qquad\square$

APPENDIX H. PROOF OF PROPOSITION 7.2

Proof. Let us define $\Psi_k(t) = \lambda_k V_k(t) + \tau_k W_k(V_k(t))$ and $\boldsymbol{\Psi}'(t) = [\Psi_1(t), ..., \Psi_k(t), ..., \Psi_m(t)]$

$E[\exp\{i\boldsymbol{s}'(\tilde{\mathbb{Y}})\}] =$

$E[\exp\{i\boldsymbol{s}'([\omega_1 V_z(t) + \tau_1 W_z(V_z(t) + \zeta(t)) + \Psi_1(t)), \ ... \ , \omega_m V_z(t) + \tau_m W_z(V_z(t) + \zeta(t)) + \Psi_m(t)]'\} =$

$E[\exp\{i\boldsymbol{s}'\boldsymbol{\omega}V_z(t) + i\boldsymbol{s}'\boldsymbol{\tau}W_z(V_z(t) + \zeta(t))\} + is\boldsymbol{\Psi}(t)]$

Let us now continue by defining the CGF for the $V_z(t)$ process as $\kappa_{V_z(t)}(u)$ and the corresponding function for the $V_k(t)$ process as $\kappa_{V_k(t)}(u)$. Both are defined using Equation 4.3.

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Let us also recall the Fourier transform for the $\zeta(t)$ process ,$\varphi_{\zeta(t)}(s)$, from Remark 7.1. Then the above expectation can be expressed as:

$E[\exp(\boldsymbol{s}'\tilde{\mathbb{Y}})(t)]$

$$= \exp\{\kappa_{V_z(t)}(-i(i\boldsymbol{s}'\boldsymbol{\omega} + \frac{(i\boldsymbol{s}'\boldsymbol{\tau})^2}{2}) + \ln\varphi_{\zeta(t)}(-i(\frac{(i\boldsymbol{s}'\boldsymbol{\tau})^2}{2}) + \sum_{k=1}^{m}\kappa_{V_k(t)}(-i(is_k\omega_k - \frac{\tau_k^2 s_k^2}{2}))\}$$

$$= \{\cosh[\frac{t}{2}g(s)] - \frac{\psi}{g(s)}\sinh[\frac{t}{2}g(s)]\}^{-\frac{2\xi}{\varsigma}}\exp\{(\delta t\gamma - \frac{\psi\xi}{\varsigma^2}) + \frac{(\boldsymbol{s}'\boldsymbol{\tau})^2\nu_0}{g(s)\coth(\frac{tg(s)}{2}) - \psi}$$

$$- \delta t[\gamma^2 - 2i\boldsymbol{s}'\boldsymbol{\omega} + (\boldsymbol{s}'\boldsymbol{\tau})^2]^{\frac{1}{2}} + Q(s)\}$$

where $Q(\boldsymbol{s}) = \sum_{k=1}^{m}\delta_k(\gamma - (\gamma^2 - 2is_k\omega_k + s_k^2\tau_k^2)^{\frac{1}{2}})$ and $g(\boldsymbol{s}) = \sqrt{\psi^2 - \varsigma^2(\boldsymbol{s}'\boldsymbol{\tau})^2}$ $\square$

Appendix I. Figures

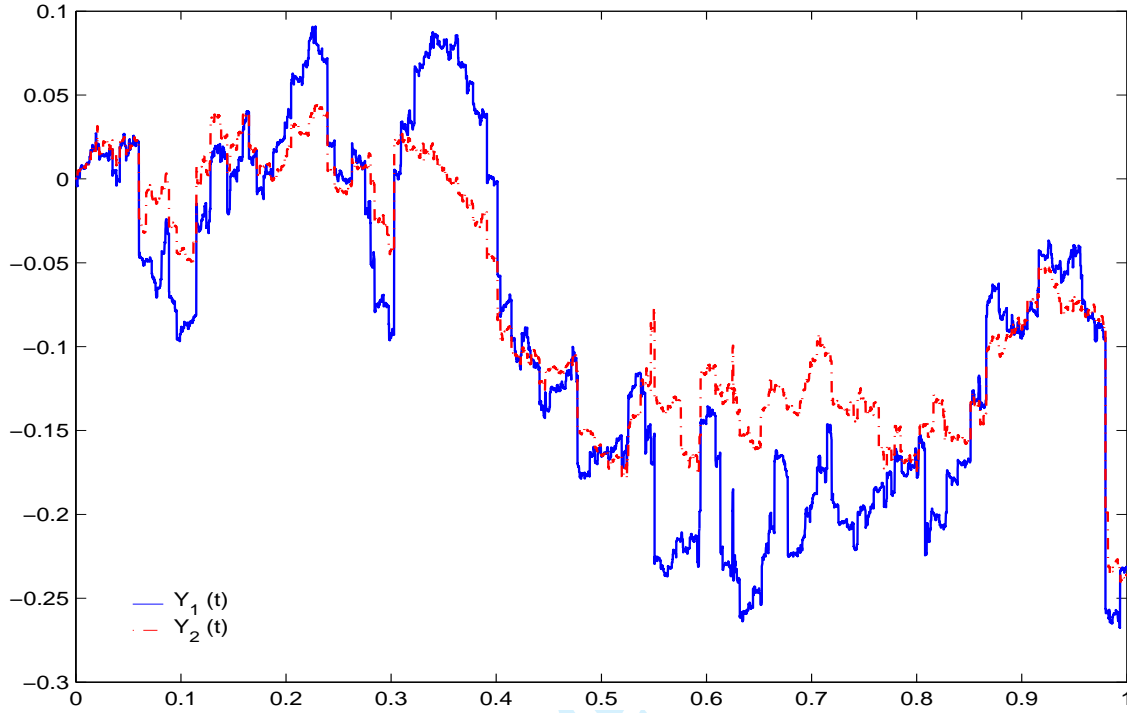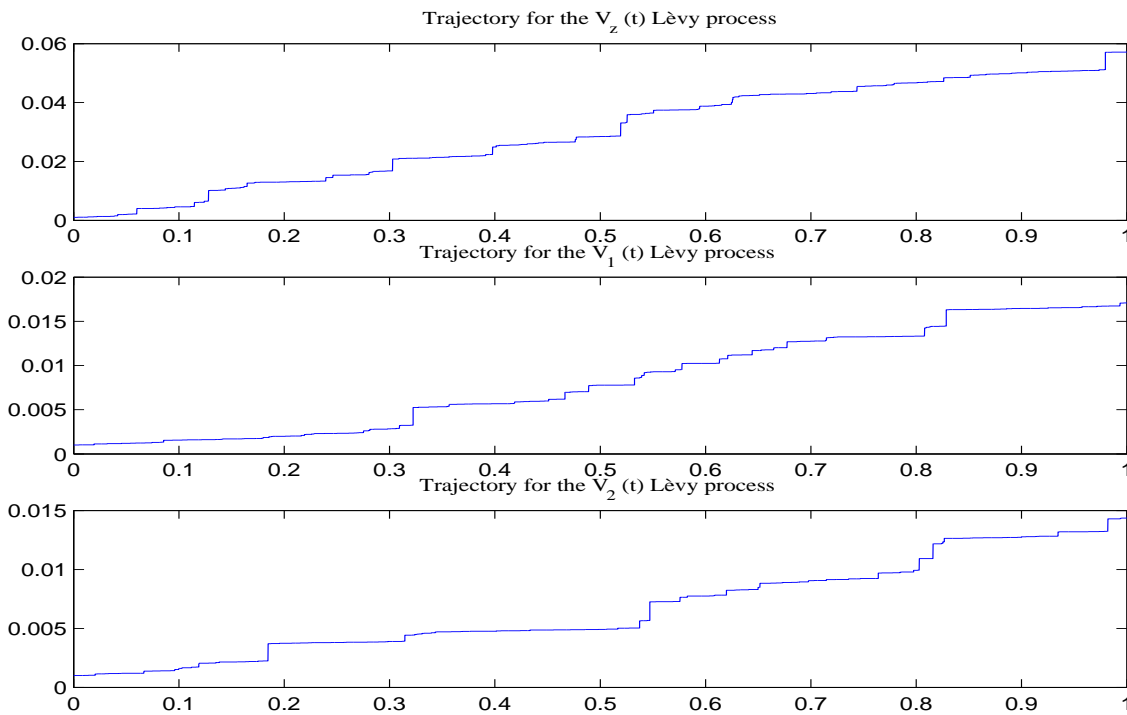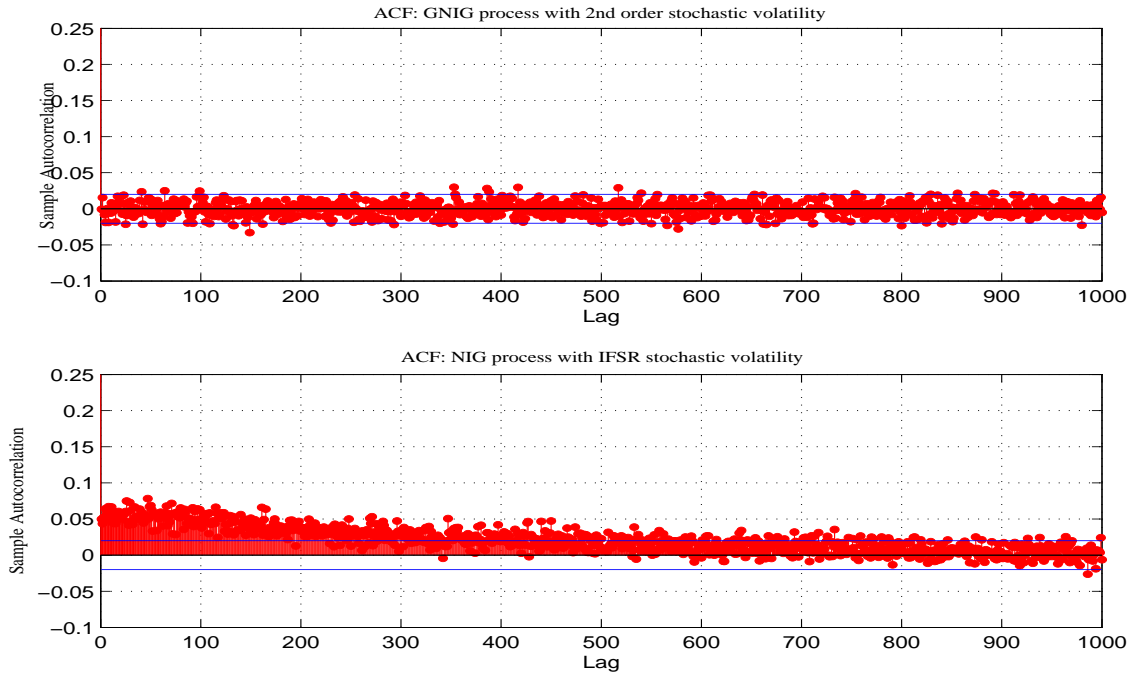*Figure 1: Trajectory bivariate GNIG Lévy process*

Figure 2: Trajectories for the individual and common subordinating processes

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Figure 3: *ACF for Stochastic Volatility and 2nd Order Stochastic Volatility*

ACF: GNIG process with 2nd order stochastic volatility

ACF: NIG process with IFSR stochastic volatility

Parameter space: $\varsigma = 1.79, \xi = 0.67, \psi = 1.21$ further $\delta = 1.00, \gamma = 16.00, \lambda = -3.20$ and $\sigma^2 = 0.8$.

Where σ^2 is only applicable for the process illustrated in the top figure.

Figure 4: *Trajectory of the stochastic volatility (with Lévy jumps) process for the log return process:*
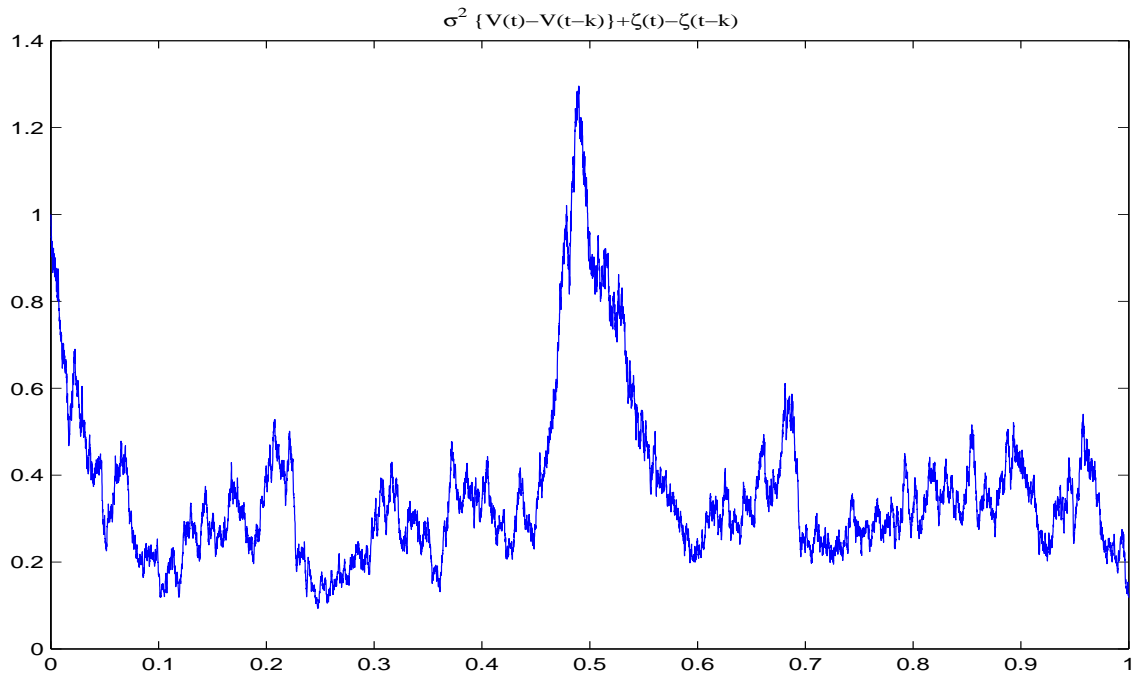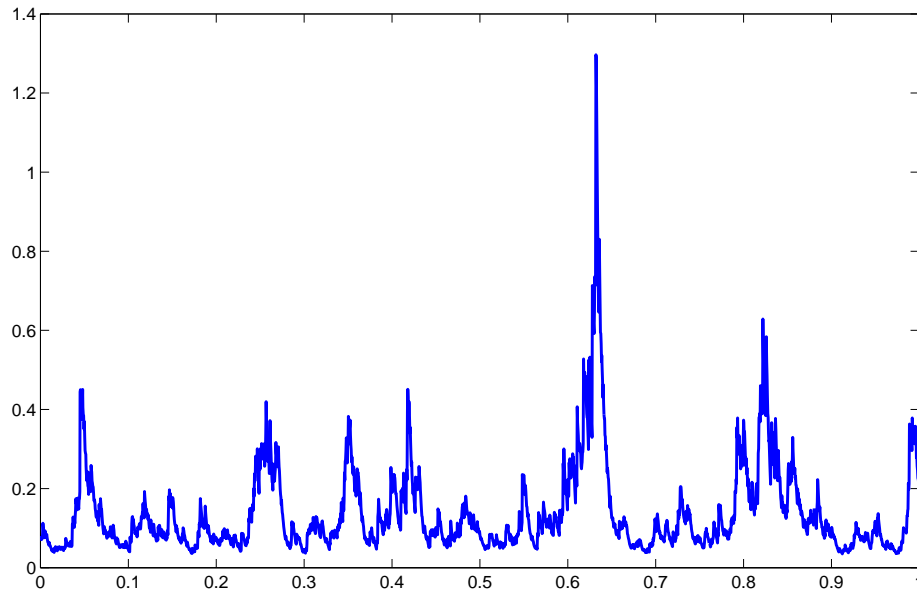

$\sigma^2 \{V(t) - V(t-k)\} + \zeta(t) - \zeta(t-k)$

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Figure 5: Trajectories for the volatility from a Student-t GARCH) process for the log return process: