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Cash Management using Multi-Stage Stochastic Programming

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Cash Management Using
Multi-Stage Stochastic Programming[*]

Robert Ferstl[†]   Alex Weissensteiner[‡]

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Abstract

We consider a cash management problem where a company with a given financial endowment and given future cash flows minimizes the Conditional Value at Risk of final wealth using a lower bound for the expected terminal wealth. We formulate the optimization problem as a multi-stage stochastic linear program (SLP). The company can choose between a riskless asset (cash), several default- and option-free bonds, and an equity investment, and rebalances the portfolio at every stage. The uncertainty faced by the company is reflected in the development of interest rates and equity returns.

Our model has two new features compared to the existing literature, which uses no-arbitrage interest rate models for the scenario generation. First, we explicitly estimate a function for the market price of risk and change the underlying probability measure. Second, we simulate scenarios for equity returns with moment-matching by an extension of the interest rate scenario tree.

Keywords: Dynamic stochastic programming, Stochastic linear programming, Cash management, Market price of risk, Change of measure, Scenario generation

1   Introduction

Treasury management refers to the set of policies, strategies, and transactions that a company adopts and implements to manage its cash resources, i.e. the investment of surpluses and the payment of liabilities in time, while taking care of the associated financial risks.

In this paper we propose a modeling strategy that can be applied to short- or medium-term cash management problems. For a given planning horizon a company tries to find an optimal investment policy for its excess cash reserves, while it has to meet future financial obligations on time. The result is a risk-optimal allocation between cash and marketable securities given a lower bound for the expected final wealth. *The uncertainty is reflected by the future development of interest rates and equity returns.* The following paragraphs give an overview of the stochastic programming literature that is related to the field of interest rate risk management.

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Bradley and Crane (1972) present a multi-stage decision model for bond portfolio management in discrete time. They develop a decomposition algorithm for linear programming, which allows an efficient, recursive solution of sub problems in the general portfolio model. Optimal global solutions are found by iterating between the sub problems and the master program. Mulvey and Zenios (1994) and Golub et al. (1995) propose to include interest rate risk scenarios by using the no-arbitrage Black-Derman-Toy (BDT) model (Black et al., 1990), which can be calibrated to the current term structure of interest rates and their volatilities. They use an option-adjusted spread to equate the market price of interest-sensitive securities with the fair price obtained by applying the expectation hypothesis. Their objective function maximizes the expected utility of final wealth.

Dupacova and Bertocchi (2001) and Bertocchi et al. (2006) solve a bond portfolio management problem with a data set of Italian government coupon bonds in the BDT context. They analyze also the sensitivity of the solution of the resulting large-scale mathematical program with respect to the model inputs and to different interval lengths between two decision stages.

Beltratti et al. (1999) and Consiglio and Zenios (2001) use the BDT interest rate model to optimize international bond portfolios. They generate scenarios using a sampling method, which assumes that exchange rates are conditional on interest rates, and that the logarithms of the ratios of interest rates and exchange rates follow a multivariate normal distribution.

Nielsen and Poulsen (2004) analyze the Danish market for mortgage backed securities. They propose a two-factor, no-arbitrage interest rate model for the scenario generation. Rasmussen and Clausen (2007) reformulate this model (again in the BDT setting) and add new features, e.g. the incorporation of fixed transaction costs and the use of scenario reduction methods. Fixed costs lead to a mixed integer problem which is numerically demanding. One of their results is that scenario reduction algorithms can generate significant arbitrage opportunities, leading to a distortion in the results.

While there is much literature on long-term asset liability management (ALM) for pension funds and insurance companies *(see e.g. Kusy and Ziemba, 1986; Gondzio and Kouwenberg, 2001; Dempster et al., 2003), only a few papers consider short-term cash or treasury management problems. Volosov et al. (2005) propose a stochastic programming model to compute currency hedging strategies and provide a backtest with a rolling horizon. Castro (2007) implements a cash management model for automatic teller machines (ATMs) and for compensation of credit card transactions.*

Many authors choose to maximize the utility of terminal wealth (see e.g. Mulvey and Zenios, 1994) or minimize a penalty function of possible shortfalls (see e.g. Dupacova and Bertocchi, 2001; Geyer and Ziemba, 2007). In this case a non-linear objective function must be linearized, raising a number of issues (see Geyer et al., 2007). In addition, it may be difficult in practice to determine the risk aversion of a company. Recently, it became popular to minimize shortfall risk measures.

In contrast to traditional bond portfolio strategies, which are based on duration matching to immunize or manage interest rate risk (see e.g. Fisher and Weil, 1971; Fong and Vasicek, 1984), modern risk measures can accommodate other risk factors such as exchange risk or equity risk. Value at Risk (VaR) is the most common one and is used by the new Basel Capital Accord. From a risk management viewpoint, VaR shows some undesirable mathematical properties, e.g. it does not satisfy the subadditivity axiom. Conditional Value at Risk (CVaR, also called "tail loss" or "expected shortfall"), as a coherent risk measure in the sense of Artzner et al. (1999), fixes these shortcomings of VaR. Pflug (2000) and Rockafellar and Uryasev (2000, 2002) show how to formulate CVaR as objective function in a linear programming framework. Recent applications of CVaR in SLP are Jobst et al. (2006) and Topaloglou et al. (2008). *Given all these*

aspects, we consider the Conditional Value at Risk to be well-suited for a cash management problem.

The contribution of this paper relative to the literature, which applies no-arbitrage interest rate models for scenario generation, is twofold: First, after using the interest rate model under the risk-neutral measure $\mathbb{Q}$ for fair bond pricing, we explicitly estimate the market price of risk from historical data to switch to the objective measure $\mathbb{P}$ for the purpose of portfolio optimization. Most of the current literature ignores this point, implicitly assuming a constant market price of risk equal to zero. This contradicts recent findings in the finance literature (see e.g. Stanton, 1997; Bernaschi et al., 2007) *where investors require a compensation for bearing (interest rate) risks.* While Jobst and Zenios (2005), Jobst et al. (2006) and Dempster et al. (2007) mention the importance of this change of measure, the present paper is the first to use an empirically estimated market price of risk to modify the conditional probabilities in the scenario tree. Second, in contrast to the current literature which uses no-arbitrage interest rate models for scenario generation, we also allow for the inclusion of an equity investment. This gives the company in our cash management problem the opportunity to reach a higher expected final wealth by taking additional risk. A large domestic index is used to represent the equity investment. We extend the scenario tree to match the first four moments of the corresponding return distribution, while taking into account the empirically observed correlation between interest rates and equity returns.

In section 2 we present the notation, the objective function, and the budget and inventory equations. Section 3 explains the scenario generation procedure, which includes the calibration of a no-arbitrage interest rate model, the estimation of the non-parametric market price of risk function, the change of measure, and the generation of equity returns. In section 4 we present a practical application, discuss the results and provide an economic interpretation of the proposed strategy. Section 5 concludes.

2 Model

We implement the cash management problem as a multi-stage stochastic linear program with recourse, using a split variable formulation of the deterministic equivalent. A general formulation of such an optimization problem is given by a linear objective function and linear constraints:

$$\min c^\top X + \mathbb{E}_\omega \, Q(X, \omega) \tag{1}$$
$$\text{s.t. } AX = b$$
$$X \geq 0$$

and

$$Q(X, \omega) = \min d(\omega)^\top Y \tag{2}$$
$$\text{s.t. } T(\omega)X + W(\omega)Y(\omega) = h(\omega),$$

where X and Y are defined as generic first- and second-stage decision variables. The linear program in (1) minimizes the first-period direct costs plus the expected recourse costs for the second-stage decisions over all the possible scenarios while meeting the first-stage constraints. The recourse function defined in (2) describes how to choose the second-stage decisions after the first-stage decision is taken and the uncertainty is revealed. It minimizes the costs subject to some recourse constraint $T(\omega)X + W(\omega)Y(\omega) = h(\omega)$.

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In our cash management application, we minimize the Conditional Value at Risk of terminal wealth under various constraints. The corresponding multi-stage decision problem is characterized by uncertainty in the interest rates and the returns of several tradeable assets. The conditional evolution of this multivariate distribution is represented in discrete time by a few mass points, which are mapped to the nodes of a so-called scenario tree in figure 1. Optimal decisions are calculated for each node of the tree.
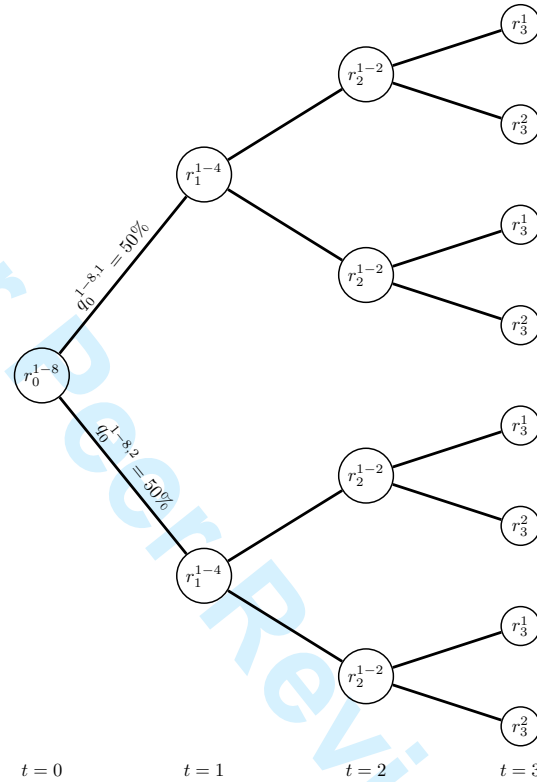
Figure 1: Scenario tree under the objective measure

The notation r_t^{i-j} refers to the interest rate valid for the period t to $t+1$ in the scenarios i to j. While the first-stage decision is unique, the recourse actions in the following stages depend on the revealed information in the scenario tree, i.e. they are stochastic decision variables.

In the following section, we introduce our notation and key variables. T is the total number of time intervals in the optimization task. The asset allocation decisions are taken at discrete time stages $t = 0, \ldots, T$, where the investor can choose between a riskless asset (cash), J different bonds and an equity investment.

2.1   List of parameters

L_t	the amount of cash outflows for liabilities at stage t; cash inflows are represented with a negative sign
S	the set of scenarios $s = 0, \ldots, S$, represented as unique paths of revealed uncertainty from the first node at stage $t = 0$ to the final nodes at stage $t = T$ in the scenario tree
r_t^s	the interest rate valid for the period t to $t+1$

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r_{et}^{s} the equity return realized at stage t

N_{t}^{s} the set of immediate successor nodes $\nu \in N_{t}^{s}$ in stage $t+1$ of the node
represented by scenario s at stage t

$\mathbf{r}(N_{t}^{s})$ the set of interest rates $r_{t}^{s\nu}$ for the subsequent stage with conditional probability
$q_{t}^{s\nu} > 0$ under the martingale measure $\mathbb{Q}$ and
$p_{t}^{s\nu} > 0$ under the objective measure $\mathbb{P}$, with $\nu \in N_{t}^{s}$

p^{s} the probability of scenario s given by $p^{s} = \prod_{t=0}^{T-1} p_{t}^{s\nu}, \quad \forall s \in S$

P_{jt}^{s} the fair price of bond j in scenario s at stage t dependent on $\mathbf{r}(N_{t}^{s})$

f_{jt}^{s} the cash flow generated under scenario s from bond j at stage t
expressed as a fraction of its face value

P_{et}^{s} the fair equity price in scenario s at stage t

$\delta_{1} \geq 0$ the spread for a positive bank account balance

$\delta_{2} \geq 0$ the spread for a negative bank account balance

tc_{b}, tc_{e} the transaction costs for purchases and sales of bonds and stocks

$\xi_{jt}^{s}, \xi_{et}^{s}$ the selling prices of bonds and stocks, i.e. $P_{jt}^{s}(1-tc_{b})$ and $P_{et}^{s}(1-tc_{e})$

$\zeta_{jt}^{s}, \zeta_{et}^{s}$ the purchasing prices of bonds and stocks, i.e. $P_{jt}^{s}(1+tc_{b})$ and $P_{et}^{s}(1+tc_{e})$

u_{e} the maximum weight of equity holdings in the total portfolio (excluding cash)

β the lower bound on expected final wealth as target value

ϕ the VaR of the optimal solution

α the critical probability level for VaR and CVaR

2.2 First-stage decision variables

b_{0} the initial cash holdings

$b_{j} \geq 0$ the initial holdings (in face value) of bond j

$b_{e} \geq 0$ the initial holdings (number) of equity shares

$x_{j0} \geq 0$ the face value of bond j purchased at stage $t=0$

$y_{j0} \geq 0$ the face value of bond j sold at stage $t=0$

$z_{j0} \geq 0$ the face value of bond j held at stage $t=0$

$x_{e0} \geq 0$ the number of equity shares purchased at stage $t=0$

$y_{e0} \geq 0$ the number of equity shares sold at stage $t=0$

$z_{e0} \geq 0$ the number of equity shares held at stage $t=0$

2.3 Second-stage decision variables

$x_{jt}^{s} \geq 0$ the face value of bond j purchased at stage t under scenario s

$y_{jt}^{s} \geq 0$ the face value of bond j sold at stage t under scenario s

$z_{jt}^{s} \geq 0$ the face value of bond j held at stage t under scenario s

$x_{et}^{s} \geq 0$ the number of stocks purchased at stage t under scenario s

$y_{et}^{s} \geq 0$ the number of stocks sold at stage t under scenario s

$z_{et}^{s} \geq 0$ the number of stocks held at stage t under scenario s

$y_{t}^{+s} \geq 0$ the lending amount at stage t under scenario s

$y_{t}^{-s} \geq 0$ the borrowing amount at stage t under scenario s

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y_T^s the balance on the bank account at final stage T under s, i.e. $y_T^{+s} - y_T^{-s}$

$\psi_T^{+s} \geq 0$ the portfolio shortfall in excess of VaR under scenario s

$\psi_T^{-s} \geq 0$ the portfolio surplus in excess of VaR under scenario s

2.4 Objective function, constraints

We minimize the Conditional Value at Risk (CVaR) of final wealth in our multi-stage cash management task. The linear programming formulation follows Pflug (2000) and Rockafellar and Uryasev (2000, 2002):

$$\phi + \frac{1}{1-\alpha} \sum_{s=1}^{S} p^s \psi_T^{+s} \to \min. \tag{3}$$

While the Value at Risk of the optimal solution is represented by ϕ, the second term accounts for the expected shortfall below the VaR for a given confidence level α.

The first-stage decision variables have to fulfill the inventory equations

$$y_{j0} + z_{j0} = b_j + x_{j0} \qquad \forall j$$
$$y_{e0} + z_{e0} = b_e + x_{e0}, \tag{4}$$

forcing the final holdings of each asset to equal the initial holdings plus purchases minus sales. This must also hold in monetary terms leading to the first-stage budget equation

$$y_0^+ + \sum_{j=1}^{J} \zeta_{j0} x_{j0} + \zeta_{e0} x_{e0} = b_0 + \sum_{j=1}^{J} \xi_{j0} y_{j0} + \xi_{e0} y_{e0}. \tag{5}$$

The variable y_0^+ denotes the lending amount after the first-stage decision, where no borrowing is allowed. Multiplying the quantities of bought and sold assets with their corresponding prices yields their transaction values. Depending on the revealed uncertainty, optimal second-stage decisions are calculated. While these variables have to fulfill the inventory equations

$$z_{jt}^s + y_{jt}^s = z_{j,t-1}^s + x_{jt}^s \qquad \forall j, s, \quad 1 \leq t \leq T$$
$$z_{et}^s + y_{et}^s = z_{e,t-1}^s + x_{et}^s \qquad \forall s, \quad 1 \leq t \leq T, \tag{6}$$

the budget equation (5) is extended for interest rate payments on the cash account, coupon payments on bond holdings and external cash outflows or inflows.

$$\sum_{j=1}^{J} \xi_{jt}^s y_{jt}^s + \xi_{et}^s y_{et}^s + \sum_{j=1}^{J} f_{jt}^s z_{j,t-1}^s + (1 - \delta_1 + r_{t-1}^s) y_{t-1}^{+s} + y_t^{-s} =$$
$$L_t + \sum_{j=1}^{J} \zeta_{jt}^s x_{jt}^s + \zeta_{et}^s x_{et}^s + (1 + \delta_2 + r_{t-1}^s) y_{t-1}^{-s} + y_t^{+s} \qquad \forall s, t \tag{7}$$

The liabilities L_t are negative for net cash inflows. Constraint (7) states that coupons are paid exactly at

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the discrete decision stages. We assume that coupon payments between two decision stages are reinvested at the current (scenario-dependent) interest rate (minus the spread δ_1) until the next stage. The maximum weight of equities in the portfolio holdings for first- and second-stage decision variables is restricted by

$$\left(z_{e0}P_{e0} + \sum_{j=1}^{J} z_{j0}P_{j0} \right) u_e \geq z_{e0}P_{e0} \tag{8}$$

$$\left(z_{et}^s P_{et}^s + \sum_{j=1}^{J} z_{jt}^s P_{jt}^s \right) u_e \geq z_{et}^s P_{et}^s \qquad \forall s, t. \tag{9}$$

On the left-hand side, the market value of the portfolio (excluding cash) is multiplied with the maximum weight of total portfolio holdings u_e. This must be greater than *or equal to* the market value of the equity investment in each stage on the right-hand side. Given the final balance on the bank account at stage T with

$$y_T^s = y_T^{+s} - y_T^{-s} \qquad \forall s, \tag{10}$$

we can calculate the portfolio shortfall in excess of Value at Risk (VaR) $\psi_T^{+s} = \max[0, -y_T^s - \phi]$. To determine the value in the maximum operator in the linear programming formulation, we introduce two non-negative auxiliary variables ψ_T^{+s} and ψ_T^{-s}

$$\psi_T^{+s} = -y_T^s - \phi + \psi_T^{-s} \qquad \forall s. \tag{11}$$

The portfolio shortfall in excess of VaR is used in the objective function (3) for the calculation of Conditional Value at Risk. We set β as a lower bound on the portfolio's expected value at the final stage T:

$$\sum_{s=1}^{S} p^s y_T^s \geq \beta. \tag{12}$$

By parametrically changing β in (12), we generate optimal solutions for different combinations of target levels of expected final wealth and Conditional Value at Risk. This allows the investor to quantify the risk-return tradeoff, and to choose a target wealth depending on the level of risk aversion.

3　Scenario generation

The uncertainty in our decision problem is represented by stochastic interest rates and equity returns. The usual approach to model a stochastic variable in continuous time is an Itô process

$$dr_t = \mu(r,t)dt + \sigma(r,t)\mathrm{d}z_t, \tag{13}$$

where the drift μ and the diffusion σ are time- and state-dependent and z_t is a standard Brownian motion. Our scenario generation procedure is based on the one-factor model for the short rate by Black et al. (1990).

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In continuous time the BDT model can be written under the martingale measure $\mathbb{Q}$ as

$$d \ln(r_t) = \left(\frac{\partial \ln(u_t)}{\partial t} - \frac{\partial \ln(\sigma_t)}{\partial t} \left(\ln(u_t) - \ln(r_t) \right) \right) dt + \sigma_t dz_t^{\mathbb{Q}}, \tag{14}$$

where u_t is the median of the short rate distribution (see Rebonato, 2002, p. 260). As can be seen from (14), the short rate is log-normally distributed, restricted to be positive, and the term structure of volatility, in contrast to a general Itô process, depends only on time. For the pricing of zero-coupon bonds and Forward Rate Agreements (FRAs) no closed-form solution exists in continuous time. Therefore, a discrete time recombining binomial tree has to be used. Due to its structure, it is well-suited to represent the uncertain scenarios in our stochastic optimization problem. In the following we present the calibration of this discrete time recombining binomial tree for the future development of r, which fits the zero coupon yield curve and the volatility curve observed on the market by implied Black volatilities of caplets with different expiry dates (see Rebonato, 2004).

3.1 Calibration of the no-arbitrage interest rate model

As input data we use EUR swap rates from the Austrian Nationalbank (OeNB) observed on October 10, 2006 and the volatility curve of at-the-money EUR interest rate caps from Tullet Prebon. We calculate the spot rate curve from the swap rates by bootstrapping, while taking care of the tenor convention in the data.[1] With these spot rates we can calculate forward rates and, since we know that the price of a cap is the sum of the prices of the individual caplets, we can use the Black market formula to strip the volatility term structure for caplets from the volatility term structure of caps (see Brigo and Mercurio, 2006). The results are given in table 1. Using these data we are able to include also the "correct" (market) expectation of the volatility curve in our tree following the algorithm proposed by Black et al. (1990). As proposed in Dempster et al. (2003), for a practical implementation we suggest to re-calibrate and re-solve the model in each future decision stage.

Table 1: Calculated EUR spot rates and caplet volatilities

Years	Spot rates	Caplet volatilities
1	3.7610	10.9493
2	3.8377	16.5826
3	3.8375	18.0779
4	3.8458	16.9128
5	3.8629	16.7248

Sources: OeNB, Bloomberg - Tullet Prebon on October 10, 2006

The resulting binomial tree of future interest rates allows for the pricing of all available coupon and zero-coupon bonds. The scenario dependent fair price of any bond P_{jt}^s is calculated as the risk-neutral expectation using backward induction in the tree, see (15). As the probabilities $q_t^{s\nu}$ for conditional successor interest rates of r_t^s are given under the martingale measure $\mathbb{Q}$, the expected value (taking conditional

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[1] The data are referenced to a six-month tenor, except for the one-year maturity, which is based on a tree-month tenor (according to the International Swap and Derivatives Association, Inc. (ISDA) and Tullet Prebon). In all cases where the tenor was shorter than the available data, we interpolated with cubic splines.

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coupon payments and future bond prices into account) can be discounted by the risk-free spot rate r_t^s.

$$P_{jt}^s = (1 + r_t^s)^{-1} \sum_{\nu \in N_s^t} q_t^{s\nu}(f_{jt}^{s\nu} + P_{jt}^{s\nu}) \tag{15}$$

Figure 2 shows the first branch of the calibrated interest rate tree along with the fair prices of the two bonds used in the practical application in section 4.
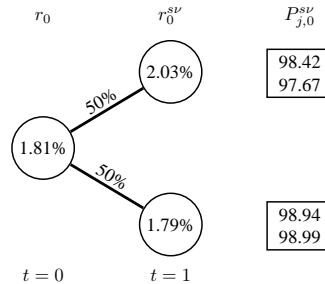

Figure 2: Sample scenario tree under the martingale measure

3.2 Market price of risk estimation

In the previous section, we have used the short interest rate process under the martingale measure $\mathbb{Q}$ for fair bond pricing. As we have to solve the portfolio optimization problem in (3) - (12) under the physical measure $\mathbb{P}$ (i.e. the real-world probabilities), we must change the probability measure from $\mathbb{Q}$ to $\mathbb{P}$ accounting for the so called *market price of risk* (MPR).

If the short rate follows an Itô process as in (13), the stochastic differential equation for a bond price with maturity T is given by

$$
\begin{aligned}
\mathrm{d}P(t,T) &= \left(\frac{\partial P}{\partial r}\mu + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \sigma \frac{\partial P}{\partial r}\mathrm{d}z_t \\
&\equiv \mu(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}z_t,
\end{aligned} \tag{16}
$$

where $\mu(t,T)$ is the drift and $\sigma(t,T)$ the diffusion term of the bond price process. By forming a riskless portfolio of two bonds with maturities T_1 and T_2, i.e. by eliminating the interest risk $\sigma(t,T)$ in the portfolio, it can be shown that the following equality must hold

$$\frac{\mu(t,T_1) - r_t P(t,T_1)}{\sigma(t,T_1)} = \frac{\mu(t,T_2) - r_t P(t,T_2)}{\sigma(t,T_2)} \tag{17}$$

for any choice of T_1 and T_2 (see Rebonato, 2002). This ratio can depend on r and t but is independent of the maturity T. In the finance literature (18) is known as the "market price of risk"

$$\lambda(r,t) = \frac{\mu(t,T) - r_t P(t,T)}{\sigma(t,T)}, \tag{18}$$

where $\mu(t,T)$ is the real-world drift and $r_t P(t,T)$ the risk-neutral drift of the price of a discount bond. By dividing numerator and denominator of the fraction in (18) by the price $P(t,T)$, the absolute bond drift

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and volatility are expressed in terms of their respective percentage quantities. The market price of risk λ represents the excess return over the riskless rate per unit volatility.

A special characteristic of bonds, seen as derivatives of the short rate, is that their underlying is not a traded asset itself. In contrast to the Black-Scholes model that uses traded stocks, it is impossible to build a replicating strategy to eliminate interest rate risk by dynamic hedging (see Björk, 2004, chap. 21) and to use this replicating portfolio for pricing.

We estimate the market price of risk in (18) as a function of the current level of the short rate r_t giving:

$$\lambda(r_t) = \frac{\mathbb{E}_t^{\mathbb{P}}\left[R^{(i)}\right] - r_t}{\sigma^{(i)}(r_t)}, \tag{19}$$

where $\mathbb{E}_t^{\mathbb{P}}\left[R^{(i)}\right]$ and $\sigma^{(i)}(r_t)$ denote the percentage quantities for the drift and the volatility of bond i as described above.[2] The resulting MPR is time homogeneous and can be used for any one-factor interest rate model. To do this we apply the approach introduced by Stanton (1997), which has the advantage that no parametric assumptions about the form of the drift and the diffusion are necessary. *His method allows to estimate the functional relationship between the market price of risk and the level of interest rates using excess returns on two Treasury bills with different maturities, see* (22). Stanton (1997) constructs approximations using Taylor series, which converge to the true functions as the time steps become smaller. He compares the performance of higher order approximations for different sampling intervals and finds that for daily data and over a wide range of values, even simple first order approximations are very accurate.

In the rest of this section we describe how to perform the kernel density estimation of the first order approximations for the drift, the diffusion, and the market price of risk. Our data set consists of daily three-month and six-month EURIBOR/FIBOR rates starting on July 2, 1990 (see figure 3).
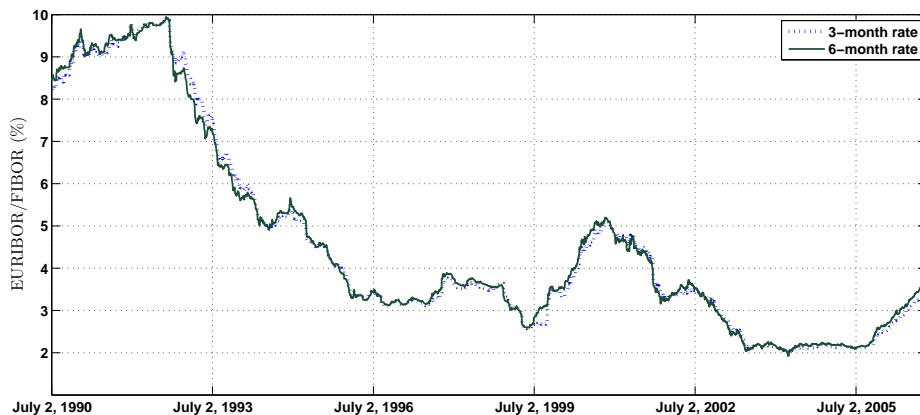

Figure 3: FIBOR/EURIBOR rates (Source: Thomson Financial DataStream)

Stanton (1997) derives the following first order approximations for the drift

$$\mu(r_t) = \frac{1}{\Delta}\mathbb{E}_t^{\mathbb{P}}\left[r_{t+\Delta} - r_t\right] + \mathcal{O}(\Delta) \tag{20}$$

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[2]For easy comparability, our notation follows that in Stanton (1997). There, the dependence of the bond's return on the short rate is also omitted for the sake of brevity.

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and the diffusion

$$\sigma(r_t) = \sqrt{\frac{1}{\Delta} \mathbb{E}_t^{\mathbb{P}} [(r_{t+\Delta} - r_t)^2]} + \mathcal{O}(\Delta), \tag{21}$$

where Δ is the time (in years) between successive observations, e.g. for daily data $\Delta = 1/250$. For the estimation of the expectation in (20) and (21) we use the three-month FIBOR/EURIBOR as the short-term riskless rate. Stanton (1997) also derives an approximation for the excess return, which we denote as $\lambda'(r_t)$. The multiplication of the right-hand side of (19) with $\sigma(r_t)$ eliminates the scaling per unit of volatility and gives the absolute excess return

$$\lambda'(r_t) = \frac{\sigma(r_t)}{\Delta(\sigma^{(1)}(r_t) - \sigma^{(2)}(r_t))} \mathbb{E}_t^{\mathbb{P}} \left[R_{t,t+\Delta}^{(1)} - R_{t,t+\Delta}^{(2)} \right] + \mathcal{O}(\Delta). \tag{22}$$

While $R_{t,t+\Delta}^{(i)}$ is the holding period return on a non-dividend paying security i, the difference in the square brackets of (22) indicates the excess return of the six-month over the three-month Treasury bill. For the calculation of the log-returns $R_{t,t+\Delta}^{(i)}$ we convert the yields in figure 3 to the corresponding prices

$$P^{(1)} = \frac{1}{1 + r_t^{(1)}(6/12)} \quad \text{and} \quad P^{(2)} = \frac{1}{1 + r_t^{(2)}(3/12)}. \tag{23}$$

The expected excess return in (22) is multiplied by a scaling factor that consists of the diffusion from (21) in the numerator and the difference of the instantaneous volatilities times Δ in the denominator. Note that dividing by the volatility difference amounts to a scaling per unit of risk similar to (19). The instantaneous volatility of asset i is given by

$$\sigma^{(i)} = \frac{P_r^{(i)} \sigma(r_t)}{P^{(i)}}, \tag{24}$$

where $P_r^{(i)}$ is the partial derivative of the price of the i-th asset with respect to the interest rate. As can be seen in (24), the instantaneous volatility is calculated by multiplying the estimated volatility in (21) with the modified duration.

We estimate the functions in (20) - (22) with a kernel density estimation procedure from the historical data shown in figure 3. The expectation in the first order approximation of the drift μ in (20) is obtained by

$$\mathbb{E}_t^{\mathbb{P}} [r_{t+\Delta} - r_t | r_t = r] \approx \frac{\sum_{n=1}^{N-1} (r_{n+\Delta} - r_n) K [(r - r_n)/h]}{\sum_{n=1}^{N-1} K [(r - r_n)/h]}, \tag{25}$$

where $K(z) = (2\pi)^{-1/2} e^{-(1/2)z^2}$ is the normal kernel function for a variable z, and $h = \hat{\sigma} N^{-1/(m+4)}$ is the window width. The total number of observations is N, $\hat{\sigma}$ denotes the estimate of the standard deviation of the three-month interest rates, and $m = 3$ is the smoothing factor. We estimate (21) and (22) in an analogous manner.

The result is the function for the market price of risk in figure 4, which is *clearly* different from zero. Negative values indicate a positive premium for taking interest rate risk.

3.3 Change of measure

The final step in our scenario generation procedure involves changing the risk-neutral probabilities $q_t^{s\nu}$ to account for the estimated market price of risk and to simulate equity returns. As described above, the BDT
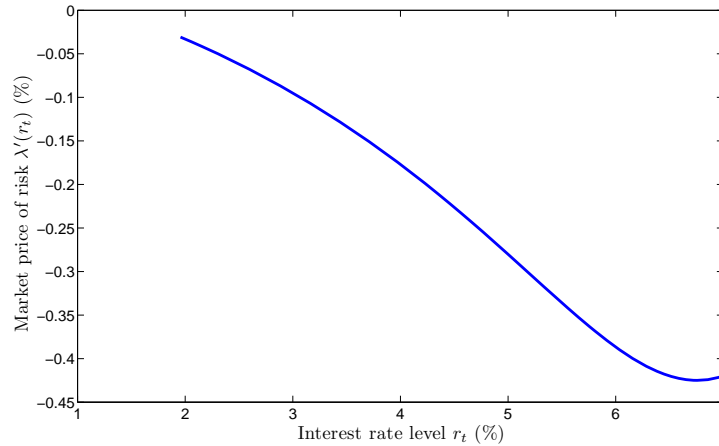
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Figure 4: Estimated market price of interest rate risk

interest rate model represents the potential evolution of future spot rates by a recombining binomial tree, where the probabilities of the two successor nodes equal 50% (see figure 2). For the purpose of portfolio optimization, we use the estimated market price of risk (see Stanton, 1997) in figure 4 to switch to the objective measure $\mathbb{P}$ by modifying the conditional probabilities of the successor nodes to $p_t^{s\nu}$ such that (26) holds:

$$\mathbb{E}^{\mathbb{Q}}(\mathbf{r}_t^{s\nu}) + \lambda'(r_t^s) = \mathbb{E}^{\mathbb{P}}(\mathbf{r}_t^{s\nu}). \tag{26}$$

The probability measure is changed in such a way that the expectation of the interest rate under $\mathbb{P}$ is equal to the expectation under $\mathbb{Q}$ plus the expected excess return from (22). The fair bond prices calculated from the calibrated interest rate tree in figure 2 are unaffected by this change of measure. Therefore, we modify only the probabilities of the interest and price scenarios while $r_0^{s\nu}$, $P_{1,0}^{s\nu}$ and $P_{2,0}^{s\nu}$ remain the same (see also figure 5).
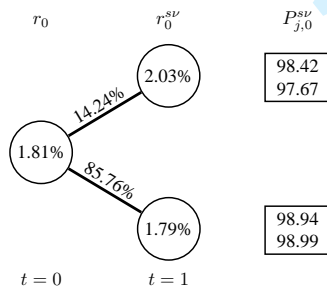

Figure 5: Sample scenario tree under the objective measure

3.4 Equity return generation

Next, we extend the scenario tree to allow for a second source of uncertainty in our model, i.e. the evolution of equity returns. They are characterized by the first four moments of their distribution and the correlation with the spot rate. The binomial structure of the scenario tree used to model the interest rate uncertainty

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leads to a lack of degrees of freedom, making it impossible to match all four moments of the equity return distribution. To resolve this shortcoming, we duplicate the successor nodes of each unique predecessor node in the original scenario tree, and the branching factor is increased to four. We use a nonlinear optimizer to change the equity returns $r_{e0}^{s\nu}$ and the probabilities $p_0^{s\nu}$ in figure 6 in such a way that they match the moments of the empirical equity return distribution and the correlation with the spot rate. Simultaneously, we force the probabilities of the doubled successor nodes (i.e. the first two and the last two nodes in figure 6) to sum up to the original probabilities in figure 5.
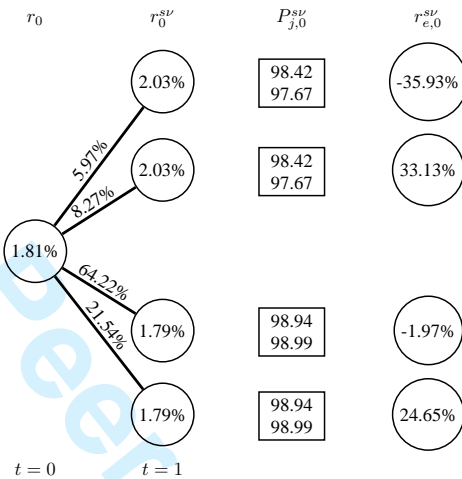

Figure 6: Sample scenario tree including simulated equity returns

We apply the procedure proposed by Klaassen (2002) *to generate equity returns in such a way that arbitrage opportunities are ruled out,* since these would be exploited by the optimization algorithm. Further, so-called "non-anticipativity constraints" are imposed to guarantee that a decision made at a specific node is identical for all scenarios leaving that node.

4 Practical application

We consider a company with a planning horizon of 2.5 years, where portfolio rebalancing is allowed semi-annually from $t = 0$ to $T = 5$. The branching factor of four and the five time intervals lead to a tree with $1,024 \, (= 4^5)$ scenarios. The valuation date is October 10, 2006 and the critical probability level α is set to 95%.

The initial cash endowment in $t = 0$ is $b_0 = 100$. No bonds and equities are held at the beginning, so that $b_j = 0 \quad \forall j$ and $b_e = 0$. The transaction costs included in the purchasing and selling prices are $tc_b = 1.00\%$ for bonds and $tc_e = 1.00\%$ for equities. Cash can be invested with a spread of $\delta_1 = 1.00\%$ and borrowed with a spread of $\delta_2 = 1.50\%$. The company has to match the liabilities L_t given in table 2. Negative values, as in stage $t = 3$, refer to net cash inflows.

Table 2: Liabilities

Stage	1	2	3	4	5
Cash flow	35	10	-7	25	40

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The available interest-sensitive assets are two default- and option-free coupon bonds as indicated in table 3. Their prices are uncertain and depend on the evolution of the future interest rates as shown in figure 5.

Table 3: Available bonds

Bonds	Coupon	Maturity	Coupon frequency
Bond 1	2.50%	April 10, 2008	semi-annually
Bond 2	3.00%	April 10, 2009	annually

Moreover, the company can invest in equities whose characteristics are given in table 4.

Table 4: Characteristics of equity returns

Stocks	
Excess return % p.a.	5.60
Volatility % p.a.	23.60
Skewness	−0.11
Kurtosis	3.22
Correlation with r	−0.01

The excess return of the equity investment is set to the average annual German short-horizon equity risk premium from Ibbotson Associates, Inc. (2006), i.e. the historical excess return over the money market rate for an investment between 1970 and 2005. The expected gross return of equities for each node in the scenario tree is calculated by adding this time-scaled risk premium to the scenario-dependent risk-free rates under the probability measure $\mathbb{P}$. In this way we can guarantee that in states of the world with high nominal interest rates, also high equity returns can be expected in our scenario tree and vice versa. The higher order moments *(volatility, skewness and kurtosis)* are estimated from semi-annual historical data of the DAX index (from January 1, 1993 to July 1, 2006). The correlation is calculated between the three-month EURIBOR/FIBOR rate and the stock index return.

We use the modeling language AMPL / GLPK to formulate the problem and solve the deterministic equivalent with CPLEX / CLP. On an Intel machine with a 2.16 GHz dual-core processor and 2 GByte RAM the calculation time for one optimization task is approximately 70 seconds when using GLPK and CLP. In our tests these open-source software packages gave the same results as their commercial counterparts (AMPL and CPLEX).

We solve the problem for a given scenario tree by gradually increasing the lower bound on the expected final wealth β. The first-stage asset allocation decisions and their corresponding wealth targets are shown in figure 7. The investor substitutes Bond 1, Bond 2 and the cash holdings with the equity investment when he requires a higher expected final wealth. Due to transaction costs and the immediate future outflows in six and twelve months, in all cases some amount of money is invested in the cash account.

Figure 8 shows the second-stage decisions of Bond 2 for a target wealth $\beta = 5$. The holdings $z_{2,t}^s$ are given as fractions of the face value set equal to 100. We want to point out that each scenario has a different probability under the measure $\mathbb{P}$ (for reasons see sections 3.3 and 3.4).
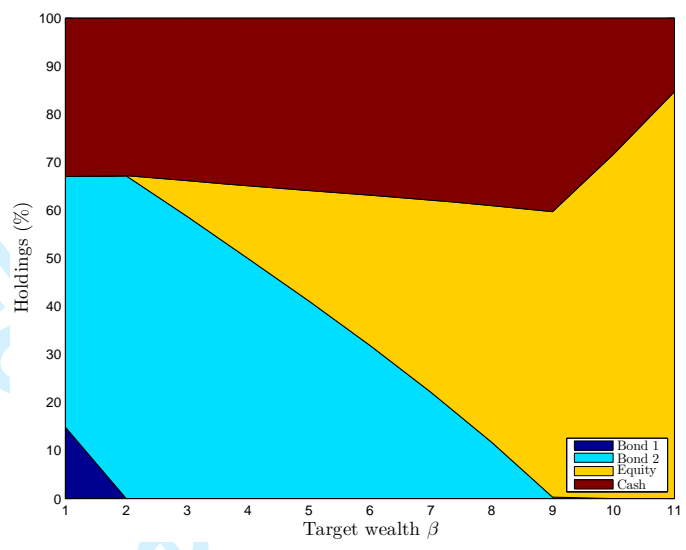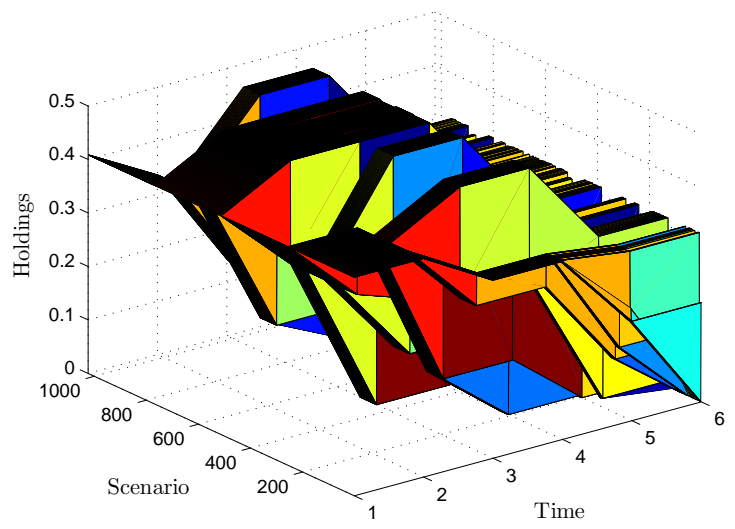
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Figure 7: First-stage asset allocation

Figure 8: Second-stage decision variables for Bond 2 ($\beta = 5$)

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In figure 9 we plot the expected target wealth β against the minimal Conditional Value at Risk (objective function). The company can choose a desired target wealth that leads to an efficient tradeoff *depending on* its risk preference.
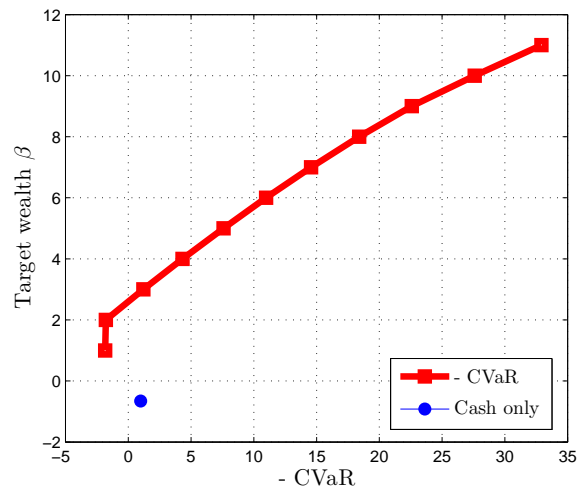

Figure 9: Risk-return tradeoff

As common in risk management, we plot the CVaR without the negative sign. The optimal risk-return tradeoff in figure 9 is compared to a naive strategy of only holding cash. We see that the cash management model generates higher expected returns for the same level of risk.

A main contribution of this paper is the estimation and inclusion of the empirically observed market price of risk in the SLP context. As a logical consequence, one might ask for the practical relevance of this adjustment. In table 5 we investigate its impact on the first-stage decisions.

Table 5: Effect of the market price of risk adjustment on first-stage asset allocation and CVaR

β	No adjustment for MPR					Adjustment for MPR				
	B1 %	B2 %	E %	C %	CVaR	B1 %	B2 %	E %	C %	CVaR
1.00	0.10	0.52	0.00	0.38	1.84	0.15	0.52	0.00	0.33	1.84
2.00	0.14	0.48	0.01	0.37	1.52	0.00	0.67	0.00	0.33	1.79
3.00	0.10	0.46	0.07	0.37	-0.99	0.00	0.59	0.07	0.34	-1.20
4.00	0.08	0.41	0.13	0.38	-3.60	0.00	0.50	0.15	0.35	-4.31
5.00	0.07	0.33	0.20	0.40	-6.34	0.00	0.41	0.23	0.36	-7.59
6.00	0.04	0.27	0.27	0.43	-9.22	0.00	0.32	0.31	0.37	-10.97
7.00	0.03	0.17	0.34	0.46	-12.27	0.00	0.22	0.40	0.38	-14.56
8.00	0.01	0.09	0.42	0.48	-15.47	0.00	0.12	0.49	0.39	-18.40
9.00	0.00	0.00	0.50	0.50	-18.96	0.00	0.00	0.59	0.40	-22.59
10.00	0.00	0.00	0.60	0.40	-22.99	0.00	0.00	0.72	0.28	-27.59
11.00	0.00	0.00	0.71	0.29	-27.24	0.00	0.00	0.85	0.15	-32.92

The market price of risk adjustment heavily influences the optimal first-stage asset allocation in our setting. We interpret the results in the following way: Including the market price of risk changes the expected returns for all assets, i.e. cash holdings, bonds and also equities. *We note two opposite effects: First, the expected spot rate becomes lower after the adjustment for the market price of risk. Second, the*

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expected return on the bonds increases. Of course this effect is stronger for Bond 2 which has the longer time to maturity. This leads to a reduction in the holdings of Bond 1 and in cash and an increase in the holdings of Bond 2. Further, the necessary holdings of equity increases in all cases, leading to a higher CVaR.

5 Conclusions

We have presented a stochastic linear programming model to solve a cash management problem. The goal is to find an optimal asset allocation decision that accounts for future uncertainty within a finite planning horizon. The objective function minimizes a coherent risk measure, i.e. the Conditional Value at Risk (CVaR) associated with final wealth. As a result, we obtain optimal decisions for the choice of cash, bonds and stocks.

The major contribution of this paper is changing the probability measure in the calibrated interest rate scenario tree. By including the market price of risk, we switch from the risk-neutral world under $\mathbb{Q}$ to the physical or objective measure $\mathbb{P}$. This transformation is necessary because the cash management optimization problem is solved in the real world, whereas the risk-free probabilities are only used for pricing the interest rate sensitive securities. Our results show a significant impact on the first-stage decision variables, but obviously all future stages are affected.

Future research could compare the benefits of the calculated strategies in the multi-stage stochastic programming context to more traditional methods, e.g. duration matching of the liabilities with tradeable bonds and without equities. Another possible extension is the inclusion of interest rate derivatives and a performance evaluation in a rolling-forward context.

Acknowledgments

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