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**Well-posedness and invariant measures for HJM models  
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# Well-posedness and invariant measures for HJM models with deterministic volatility and Lévy noise

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## Abstract

We give sufficient conditions for existence, uniqueness and ergodicity of invariant measures for Musiela's stochastic partial differential equation with deterministic volatility and a Hilbert space valued driving Lévy noise. Conditions for the absence of arbitrage and for the existence of mild solutions are also discussed.

*Keywords:* HJM models, Musiela's stochastic PDE, invariant measures

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## 1 Introduction

The aim of this work is to study some asymptotic properties of a Heath-Jarrow-Morton model of the term structure of interest rates driven by an infinite dimensional Lévy noise. In particular, denoting by  $u(t, x)$ ,  $t, x \geq 0$ , the forward rate at time  $t$  with maturity  $t + x$ , we shall be concerned with the following stochastic partial differential equation (SPDE):

$$du(t, x) = [u_x(t, x) + f(t, x)] dt + \langle \sigma(t, x), dY_0(t) \rangle, \quad (1.1)$$

where  $Y_0$  is a Lévy process taking values in a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,  $\sigma$  is a deterministic volatility term, and  $f$  is such that discounted prices of zero-coupon bonds are local martingales. Precise assumptions will be stated below. Note that in the special case where  $Y_0$  is a Wiener process, (1.1) can be written in the more familiar form

$$du(t, x) = [u_x(t, x) + f(t, x)] dt + \sum_{k=1}^{\infty} \sigma^k(t, x) dw_k(t), \quad (1.2)$$

where  $w_k$  are real independent Wiener processes and  $f$  satisfies the well-known HJM drift condition ([15], [10], [14])

$$f(t, x) = \sum_{k=1}^{\infty} \sigma^k(t, x) \int_0^x \sigma^k(t, y) dy. \quad (1.3)$$

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Invariant measures and the asymptotic behavior of (1.2), in the time-homogeneous case, have been studied by several authors (see e.g. [23], [30], [29]), allowing also the volatility coefficient to depend on the forward rate itself. Indeed it is widely accepted that mean reversion is a characteristic property of the dynamics of interest rates, and it is supported by empirical findings.

On the other hand, the literature on HJM models driven by Lévy processes has considerably grown in the last few years: let us just cite, among others, [1], [8], and of course the work from where it all started [2] (where general random measures are added to Brownian motion as driving noises). The asymptotic behavior of such models, however, does not seem to have been addressed. The present paper offers a first step in this direction, in a simple model with deterministic volatility. The setting is similar to that of [30] (where the case of HJM models driven by Wiener process was considered), but the choice of state space is different, and the well-posedness of the model is somewhat more complicated.

The paper is organized as follows: in section 2 we derive sufficient conditions on  $f$  ensuring that the bond market is arbitrage free and write the SPDE (1.1) as an abstract evolution equation in a suitable Hilbert space of forward curves, about which we discuss existence and uniqueness of mild and weak solutions. In section 3 we discuss existence and uniqueness of invariant measures, as well as the convergence in law of forward curves as time goes to infinity.

After the first draft of this paper was completed, the author was informed of some papers (most of which not yet published nor posted to any standard preprint server) with some overlap with the present one. In particular, drift conditions were derived in [24] and in [17]. Our derivation, as well as the setup, is slightly different, and we include it for the sake of completeness. Existence and uniqueness of local mild solutions for HJM models driven by finitely many independent Lévy processes is discussed in [11] (see also [21]), where the volatility is also allowed to be state dependent. In the present paper we consider a possibly infinite dimensional driving Lévy process, which is not supposed to be the superposition of independent one-dimensional Lévy processes, and we look for global solutions. Moreover, in [25] the authors obtain conditions for existence and uniqueness of global mild solutions with an infinite dimensional Lévy process and state dependent volatility. However, their state space is a weighted  $L_2$  space, which seems inappropriate for modeling purposes: in fact very irregular forward curves cannot be excluded. We would also like to point out that the choice of state space in [11] is the same as in our paper, and for this reason the former paper only treats local solutions. On the other hand, as already observed (see also [10] for an extensive discussion), our choice of state space is better than that in [25]. Finally, the existence and uniqueness statement of proposition 8 can be deduced by the result in [27] (see also [28] for a result on ergodicity of HJM models with one-dimensional, possibly multiplicative noise, in spaces of weighted  $L_2$  curves).

Let us conclude introducing some notation. Given two separable Hilbert spaces  $H$ ,  $K$  we shall denote by  $\mathcal{L}(H, K)$ ,  $\mathcal{L}_1(H, K)$  and  $\mathcal{L}_2(H, K)$  the space of bounded linear, trace-class, and Hilbert-Schmidt operators, respectively, from  $H$  to  $K$ .  $\mathcal{L}_1^+$  stands for the subset of  $\mathcal{L}_1$  of positive operators. We shall write  $\mathcal{L}_1(H)$  in place of  $\mathcal{L}_1(H, H)$ , and similarly for the other spaces. Given a self-adjoint operator  $Q \in \mathcal{L}_1^+(H)$ , we set  $|x|_Q^2 := \langle Qx, x \rangle$ ,  $x \in H$ . The Hilbert-Schmidt norm is denoted by  $|\cdot|_2$ . The characteristic function of a set  $A$  is denoted by  $\chi_A$ , and  $\chi_r$  stands for the characteristic function of

the set  $B_r := \{x \in H : |x| \leq r\}$ , where  $H$  is a Hilbert space. Given a continuously differentiable increasing function  $\alpha : \mathbb{R}_+ \rightarrow [1, \infty)$  such that  $\alpha^{-1/3} \in L_1(\mathbb{R}_+)$ , we define  $L_{2,\alpha}^n := L_{2,\alpha}^n(\mathbb{R}_+)$  as the space of distributions  $\phi$  on  $\mathbb{R}_+$  such that  $\int_0^\infty |\phi^{(n)}(x)|^2 \alpha(x) dx < \infty$ .

## 2 Musiela's SPDE with Lévy noise

Throughout the paper  $Y_0(t)$ ,  $t \geq 0$ , shall denote a Lévy process taking values in a (fixed) Hilbert space  $K$ , with generating triplet  $[b_0, R_0, m_0]$ , i.e.

$$\log \mathbb{E} e^{i\langle y, Y_0(1) \rangle} = i\langle b_0, y \rangle - \frac{1}{2} \langle R_0 y, y \rangle + \int_H (e^{i\langle \xi, y \rangle} - 1 - i\langle \xi, y \rangle \chi_1(\xi)) m_0(d\xi),$$

with  $b_0 \in K$ ,  $R_0 \in \mathcal{L}_1^+(K)$ , and  $m_0$  a  $\sigma$ -finite measure on  $\mathcal{B}(K)$ , the Borel  $\sigma$ -algebra of  $K$ , satisfying

$$m_0(\{0\}) = 0, \quad \int_K |\xi|^2 m_0(d\xi) < \infty$$

(see e.g. [13] for details). This integrability assumption serves two purposes: it ensures that  $\mathbb{E}|Y_0(t)|^2$  is finite for all  $t \geq 0$ , thus allowing to construct mild solutions of the SPDE (1.1) via an  $L_2$  theory of stochastic integration, and it allows to use Fubini's theorem to establish no-arbitrage sufficient conditions. The assumption will turn out not to be a real restriction, as the no-arbitrage condition essentially requires existence of exponential moments of  $m_0$  (see also [1]).

### 2.1 Drift condition

Let us consider, in the spirit of the original paper [15] (see also [14]), the following integral equation

$$u(t, x) = u(0, x) + \int_0^t b(s, x) ds + \int_0^t \langle \sigma(s, x), dY_0(s) \rangle, \quad (2.1)$$

where  $b$  and  $\sigma$  are random vector fields predictable in  $s$  and Borel measurable in  $x$ . In particular, they could depend on  $u$  itself.

We shall give conditions under which the dynamics (2.1) is compatible with a no-arbitrage hypothesis, namely that the corresponding discounted bond prices are local martingales. The arguments of the proof consist of the Lévy-Itô decomposition in Hilbert space (see e.g. [7]) and the calculus for random measures, following [2]. We denote by  $\hat{P}(t, \tau)$ ,  $0 \leq t \leq \tau$ , the discounted price of a zero-coupon bond expiring at time  $\tau$ .

**Theorem 1.** *Let  $\Sigma(t, x) = -\int_0^x \sigma(t, y) dy$  and  $E(t, x, \xi) = -\int_0^x \langle \sigma(t, y), \xi \rangle dy$  for all  $x \geq 0$ ,  $\xi \in K$ . Assume that for all  $t < \infty$*

$$\int_0^t \int_0^\infty |b(s, x)| dx ds < \infty, \quad \int_0^t \int_0^\infty |\sigma(s, x)|^2 dx ds < \infty. \quad (2.2)$$

Moreover, assume that for all  $x \geq 0$  one has

$$\begin{aligned} \int_0^x b(t, y) dy &= u(t, x) - u(t, 0) + \frac{1}{2} |\Sigma(t, x)|_{R_0}^2 \\ &+ \int_H \left( e^{E(t, x, \xi)} - 1 - E(t, x, \xi) \chi_1(\xi) \right) m_0(d\xi) \end{aligned} \quad (2.3)$$

$d\mathbb{P} \times dt$ -a.e. Then the discounted bond price process  $t \mapsto \hat{P}(t, \tau)$ ,  $t \geq 0$ , is a local martingale for all  $\tau \geq t$ .

*Proof.* The Lévy process  $Y_0$  admits the decomposition

$$Y_0(t) = b_0 t + W(t) + \int_0^t \int_{|\xi| < 1} \xi \tilde{N}(ds, d\xi) + \int_0^t \int_{|\xi| \geq 1} \xi N(ds, d\xi), \quad (2.4)$$

where  $b_0 \in K$ ,  $W$  is a  $K$ -valued Wiener process with covariance operator  $R_0$ ,  $N$  is a Poisson measure on  $K$  with compensator  $m_0$ , and  $\tilde{N}(ds, d\xi) := N(ds, d\xi) - ds m_0(d\xi)$ .

Using the decomposition (2.4) one can write

$$\begin{aligned} u(t, x) &= u(0, x) + \int_0^t b(s, x) ds + \int_0^t \langle \sigma(s, x), dW(s) \rangle \\ &\quad + \int_0^t \langle \sigma(s, x), d\tilde{z}(s) \rangle + \int_0^t \langle \sigma(s, x), dz(s) \rangle, \end{aligned}$$

where

$$\tilde{z}(t) = \int_0^t \int_{|\xi| < 1} \xi \tilde{N}(ds, d\xi), \quad z(t) = \int_0^t \int_{|\xi| \geq 1} \xi N(ds, d\xi).$$

However, let  $(e_k)_{k \in \mathbb{N}}$  be a base of  $K$  and set  $x^k = \langle x, e_k \rangle$ ,  $x \in K$ . Then one has

$$\begin{aligned} \int_0^t \langle \sigma(s, x), d\tilde{z}(s) \rangle &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_0^t \sigma^k(s, x) d\tilde{z}^k(s) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_0^t \int_{|\xi| < 1} \sigma^k(s, x) \xi^k \tilde{N}(ds, d\xi) \\ &= \lim_{n \rightarrow \infty} \int_0^t \int_{|\xi| < 1} \sum_{k=1}^n \sigma^k(s, x) \xi^k \tilde{N}(ds, d\xi), \end{aligned} \quad (2.5)$$

where in the second line we have used the associativity of stochastic integrals (i.e., using the notation of the stochastic calculus of semimartingales and stochastic measures,  $H \cdot (K * \mu) = (HK) * \mu$  – see e.g. [16]). Since  $\sum_{k=1}^n \sigma^k(s, x) \xi^k$  converges (for any fixed  $x$ ) to  $\langle \sigma(s, x), \xi \rangle$  in  $L_2(\Omega \times [0, t] \times H, \mathcal{P}, d\mathbb{P} \times ds \times dm_0)$  for all  $t \geq 0$ , with  $\mathcal{P}$  the predictable  $\sigma$ -field, one finally obtains

$$\int_0^t \langle \sigma(s, x), d\tilde{z}(s) \rangle = \int_0^t \int_{|\xi| < 1} \eta(s, x, \xi) \tilde{N}(ds, d\xi),$$

where  $\eta(s, x, \xi) = \langle \sigma(s, x), \xi \rangle$ . Similar reasoning shows that the same type of identity holds for integrals with respect to  $N$ .

For  $x \geq 0$ , let us denote by  $p(t, x)$  the discounted prices at time  $t$  of a risk free zero-coupon bond expiring at time  $t + x$ . By the definition of  $p(t, x)$  and the equation for  $u$

we obtain

$$\begin{aligned}
\log p(t, x) &= - \int_0^t u(s, 0) ds - \int_0^x u(t, y) dy \\
&= - \int_0^t u(s, 0) ds - \int_0^x u(0, y) dy \\
&\quad - \int_0^x \int_0^t b(s, y) ds dy - \int_0^x \int_0^t \langle \sigma(s, y), dW(s) \rangle dy \\
&\quad - \int_0^x \int_0^t \int_{|\xi| < 1} \eta(s, y, \xi) \tilde{N}(ds, d\xi) dy - \int_0^x \int_0^t \int_{|\xi| \geq 1} \eta(s, y, \xi) N(ds, d\xi) dy \\
&= - \int_0^t u(s, 0) ds - \int_0^x u(0, y) dy + \int_0^t B(s, x) ds + \int_0^t \langle \Sigma(s, x), dW(s) \rangle \\
&\quad + \int_0^t \int_{|\xi| < 1} E(s, x, \xi) \tilde{N}(ds, d\xi) + \int_0^t \int_{|\xi| \geq 1} E(s, x, \xi) N(ds, d\xi),
\end{aligned}$$

where the third equality follows by Fubini's theorem (see [5] and [18]) and the definitions of  $\Sigma$  and  $E$ , together with  $B(t, x) := - \int_0^t b(t, y) dy$ .

Applying Itô's formula, setting  $\zeta(t, x) = \log p(t, x)$ , one gets

$$\begin{aligned}
p(t, x) &= e^{\zeta(t, x)} = e^{\zeta(0, x)} + \int_0^t e^{\zeta(s-, x)} \left( -u(s, 0) ds + B(s, x) ds + \langle \Sigma(s, x), dW(s) \rangle \right) \\
&\quad + \frac{1}{2} \int_0^t e^{\zeta(s-, x)} d[\zeta, \zeta]^c(s, x) + \int_0^t \int_{|\xi| < 1} (e^{\zeta(s-, x) + E(s, x, \xi)} - e^{\zeta(s-, x)}) \tilde{N}(ds, d\xi) \\
&\quad + \int_0^t \int_{|\xi| \geq 1} (e^{\zeta(s-, x) + E(s, x, \xi)} - e^{\zeta(s-, x)}) N(ds, d\xi) \\
&\quad + \int_0^t \int_{|\xi| < 1} (e^{\zeta(s-, x) + E(s, x, \xi)} - e^{\zeta(s-, x)} - E(s, x, \xi) e^{\zeta(s-, x)}) m_0(d\xi) ds,
\end{aligned}$$

or equivalently

$$\begin{aligned}
\frac{dp(t, x)}{p(t-, x)} &= \left( -u(t, 0) + B(t, x) + \frac{1}{2} |\Sigma(t, x)|_{R_0}^2 \right) dt + \langle \Sigma(t, x), dW(t) \rangle \\
&\quad + \int_{|\xi| < 1} (e^{E(t, x, \xi)} - 1) \tilde{N}(dt, d\xi) \\
&\quad + \int_{|\xi| \geq 1} (e^{E(t, x, \xi)} - 1) N(dt, d\xi) \\
&\quad + \int_{|\xi| < 1} (e^{E(t, x, \xi)} - 1 - E(t, x, \xi)) m_0(d\xi) dt.
\end{aligned}$$

For  $\tau \geq t$ , by  $\hat{P}(t, \tau) = p(t, \tau - t)$  it follows that  $d\hat{P}(t, \tau) = dp(t, \tau - t) - p_x(t, \tau - t)$ , and  $p(t, \tau - t) = e^{-\int_0^t u(s, 0) ds} P(t, \tau - t)$  implies  $p_x(t, \tau - t) = e^{-\int_0^t u(s, 0) ds} P_x(t, \tau - t)$ . Setting  $P(t, \tau) = e^{-\int_0^\tau u(t, y) dy}$ , one has  $P_x(t, \tau - t) = -P(t, \tau - t)u(t, \tau - t)$  and  $p_x(t, \tau - t) = -u(t, \tau - t)p(t, \tau - t)$ , and finally

$$d\hat{P}(t, \tau) = dp(t, \tau - t) + u(t, \tau - t)\hat{P}(t, \tau) dt.$$

Together with the equation for  $p(t, x)$ , this implies

$$\begin{aligned} \frac{d\hat{P}(t, \tau)}{\hat{P}(t-, \tau)} &= \left( -u(t, 0) + u(t, \tau - t) + B(t, \tau - t) + \frac{1}{2} |\Sigma(t, \tau - t)|_{R_0}^2 \right) dt \\ &\quad + \langle \Sigma(t, \tau - t), dW(t) \rangle + \int_{|\xi| < 1} (e^{E(t, \tau - t, \xi)} - 1) \tilde{N}(dt, d\xi) \\ &\quad + \int_{|\xi| \geq 1} (e^{E(t, \tau - t, \xi)} - 1) N(dt, d\xi) \\ &\quad + \int_{|\xi| < 1} (e^{E(t, \tau - t, \xi)} - 1 - E(t, \tau - t, \xi)) m_0(d\xi) dt. \end{aligned}$$

Lightening notation a bit, one can write

$$\int_{|\xi| \geq 1} (e^E - 1) dN = \int_{|\xi| \geq 1} (e^E - 1) d\tilde{N} + \int_{|\xi| \geq 1} (e^E - 1) dm_0 dt,$$

hence  $\hat{P}(t, \tau)$  is a local martingale if

$$\begin{aligned} 0 &= -u(t, 0) + u(t, \tau - t) + B(t, \tau - t) + \frac{1}{2} |\Sigma(t, \tau - t)|_{R_0}^2 \\ &\quad + \int_{|\xi| \geq 1} (e^{E(t, \tau - t, \xi)} - 1) m_0(d\xi) + \int_{|\xi| < 1} (e^{E(t, \tau - t, \xi)} - 1 - E(t, \tau - t, \xi)) m_0(d\xi) \\ &= -u(t, 0) + u(t, \tau - t) + B(t, \tau - t) + \frac{1}{2} |\Sigma(t, \tau - t)|_{R_0}^2 \\ &\quad + \int_K (e^{E(t, \tau - t, \xi)} - 1 - E(t, \tau - t, \xi) \chi_1(\xi)) m_0(d\xi), \end{aligned}$$

and the theorem is proved.  $\square$

*Remark 2.* The above theorem implies a “drift condition” that generalizes the HJM condition (1.3). In particular, assume that  $Y(1)$  admits exponential moments, or equivalently that  $\int_H e^{z \cdot \xi} m_0(d\xi) < \infty$  for all  $z \in H$ , and define the function  $\psi(z) = \log \mathbb{E} e^{z \cdot Y(1)}$ . Then taking into account (1.1) and (2.1), (2.3) implies the following relation between the drift and the volatility functions:

$$\int_0^\cdot f(t, y) dy = \psi \left( - \int_0^\cdot \sigma(t, y) dy \right) \quad (2.6)$$

for all  $t \in [0, T]$ . Unfortunately this identity is “implicit”, and only under further assumptions can it be made more explicit (see below).

## 2.2 Abstract setting and well-posedness

We shall rewrite the SPDE (1.1) as an abstract stochastic differential equation in the space  $H = L_{2, \alpha}^1$ . The space  $H$  endowed with the inner product

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}_+} \phi'(x) \psi'(x) \alpha(x) dx + \lim_{x \rightarrow \infty} \phi(x) \psi(x)$$



is a separable Hilbert space. This choice of state space is standard and is apparently due to Filipović [10]. Nonetheless, other authors have studied related SPDEs in different function spaces, e.g. in weighted  $L_2$  spaces, weighted Sobolev spaces, or fractional Sobolev spaces (see [14], [30], [9] respectively).

Let us define on  $H$  the operator  $A : f \mapsto f'$ , with domain  $D(A) = L^1_{2,\alpha} \cap L^2_{2,\alpha}$ , which generates the semigroup of right shifts  $[e^{tA}\phi](x) := \phi(x+t)$ ,  $t \geq 0$ . Musiela's SPDE (1.1) can be written in abstract form as

$$du(t) = (Au(t) + f(t)) dt + B(t) dY_0(t), \quad (2.7)$$

where  $f(t) \equiv f(t, \cdot)$  and  $B(t) \in \mathcal{L}(K, H)$  is defined by  $[B(t)u](\cdot) = \langle \sigma(t, \cdot), u \rangle_K$ , with suitable regularity assumptions on  $\sigma$ .

Several papers deal with the solution of this type of equations in the time-independent case with  $f(t) \equiv 0$ ,  $B(t) \equiv B$ . Here we limit ourselves to mention [3], which is probably the first paper considering weak solutions (in the sense of PDEs), and [12], where an analytic approach is used to solve, even in the strong sense, equations of the type (3.1), possibly in a larger space, allowing the characteristic function of  $Y_0(1)$  to be only continuous instead of Sazonov continuous (see also [19], [20], [26]).

Our goal in this subsection is less ambitious, namely we shall only prove that the formal solution of (2.7) given by the variation of constants formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s) ds + \int_0^t e^{(t-s)A}B(s) dY_0(s) \quad (2.8)$$

is a well defined process and provides the unique weak solution to (2.7). In particular, this implies that the Lévy-based model (1.1) for the evolution of forward curves is well posed under appropriate assumptions on  $\sigma$ .

In analogy to [4], we shall prove that (2.8) belongs to the space  $\mathcal{H}_2(T)$  of mean square continuous process on  $[0, T]$  with values in  $H$  adapted to the filtration generated by  $Y$ . We endow  $\mathcal{H}_2(T)$  with the norm defined by

$$\|F\|_2^2 := \sup_{t \in [0, T]} \mathbb{E}|F(t)|_H^2.$$

**Proposition 3.** *Assume that*

$$\int_0^T (|f(t)|_H + |B(t)|_2^2) dt < \infty.$$

*Then  $u$  defined as in (2.8) belongs to  $\mathcal{H}_2(T)$ .*

*Proof.* Adaptedness is immediate by definition. Using the Lévy-Itô decomposition in the form

$$Y_0(t) = at + W(t) + \int_0^t \int_K \xi \tilde{N}(ds, d\xi),$$

with  $a = b_0 + \int_{|\xi| \geq 1} \xi m_0(d\xi)$ , and taking into account Propositions 2.2, 2.3 in [4], it is enough to consider the case when  $Y_0(t)$  has no drift and no Brownian component. In particular one has

$$\mathbb{E} \left| \int_0^t e^{(t-s)A} B(s) \int_K \xi \tilde{N}(ds, d\xi) \right|^2 \leq N^2 \int_0^T |B(t)|_2^2 \int_K |\xi|^2 m_0(d\xi) dt < \infty,$$

with  $N = \sup_{t \in [0, T]} |e^{tA}|$ . Moreover  $|\int_0^t e^{(t-s)A} f(s) ds| \leq N \int_0^T |f(s)| ds < \infty$ , hence  $\|u\|_2 < \infty$ .

Let us now prove that  $t \mapsto \mathbb{E}|u(t)|^2$  is continuous. Setting  $Y_A(t) := \int_0^t e^{(t-s)A} B(r) dY_0(r)$ , it is enough to prove that  $t \mapsto \mathbb{E}|Y_A(t)|^2$  is continuous. For  $0 \leq s \leq t \leq T$ , write

$$Y_A(t) - Y_A(s) = \int_0^s (e^{(t-r)A} - e^{(s-r)A}) B(r) dY_0(r) + \int_s^t e^{(t-r)A} B(r) dY_0(r),$$

where the two terms on the right-hand side are uncorrelated. Since

$$\begin{aligned} \mathbb{E} \left| \int_0^s (e^{(t-r)A} - e^{(s-r)A}) B(r) dY_0(r) \right|^2 &\leq \int_0^s |e^{(t-r)A} - e^{(s-r)A}|^2 |B(r)|_2^2 dr \int_K |\xi|^2 m_0(d\xi) \rightarrow 0, \\ \mathbb{E} \left| \int_s^t e^{(t-r)A} B(r) dY_0(r) \right|^2 &\leq \int_s^t |e^{(t-r)A}|^2 |B(r)|_2^2 dr \int_K |\xi|^2 m_0(d\xi) \rightarrow 0 \end{aligned}$$

as  $s \uparrow t$ , the result follows. The case  $0 \leq t \leq s \leq T$  and  $s \downarrow t$  is completely analogous, hence omitted.  $\square$

The following proposition shows that existence in the mild sense for equation (2.7) implies existence and uniqueness in the weak sense. This fact was essentially proved by A. Chojnowska-Michalik in [3] in the time-independent case. Here we give a more direct proof that closely follows [5].

**Proposition 4.** *Equation (2.7) has a unique weak solution given by (2.8).*

*Proof.* Let us define the additive process  $Y(t) = f(t) + \int_0^t B(s) dY_0(s)$  and write (2.7) in the form

$$du(t) = Au(t) dt + dY(t), \quad u(0) = u_0.$$

It is enough to consider the case  $u_0 = 0$ , the extension to the general case being immediate. We need to prove that, for  $v \in D(A^*)$ ,

$$\left\langle \int_0^t e^{(t-s)A} dY(s), v \right\rangle = \int_0^t \left\langle \int_0^s e^{(s-r)A} dY(r), A^* v \right\rangle ds + \langle Y(t), v \rangle,$$

or equivalently

$$\left\langle \int_0^t (e^{(t-s)A} - I) dY(s), v \right\rangle = \int_0^t \left\langle \int_0^s e^{(s-r)A} dY(r), A^* v \right\rangle ds.$$

In fact, the right-hand side of the previous expression can be equivalently written as

$$\int_0^t \int_0^s \langle e^{(s-r)A^*} A^* v, dY(r) \rangle ds,$$

hence, using the identity

$$\int_0^{t-s} e^{rA^*} A^* v dr = (e^{(t-s)A^*} - I)v,$$

Fubini's theorem yields the desired conclusion. Uniqueness will follow if we prove that a weak solution is a mild solution. One immediately recognizes that the proof of Lemma 5.5 in [5], repeated word by word, yields the identity

$$\langle w(t), \phi(t) \rangle = \int_0^t \langle w(s), \phi'(s) + A^* \phi(s) \rangle ds + \int_0^t \langle \phi(s), dY(s) \rangle, \quad t \in [0, T],$$

where  $w$  is a weak solution of (3.1) and  $\phi \in C^1([0, T], D(A^*))$ . Taking  $\phi(s) = e^{(t-s)A^*} \phi_0$ ,  $\phi_0 \in D(A^*)$ , implies

$$\langle w(t), \phi_0 \rangle = \left\langle \int_0^t e^{(t-s)A} dY(s), \phi_0 \right\rangle,$$

hence  $u(t) = w(t)$  because  $D(A^*) \subset H$  densely.  $\square$

It is clear that in order to obtain results on the well-posedness of the Musiela's SPDE (1.1) it is necessary to establish conditions on  $\sigma$  and  $Y_0$  such that the hypotheses of Proposition 3 are satisfied. The following sufficient conditions are obviously the most general ones, but also the hardest to verify.

**Proposition 5.** *Let  $(e_k)_{k \in \mathbb{N}}$  be a basis of  $K$ , and define  $\sigma^k(t, x) = \langle \sigma(t, x), e_k \rangle_K$ . Assume that*

$$\int_0^T \sum_{k=1}^{\infty} \int_0^{\infty} |\sigma_x^k(t, x)|^2 \alpha(x) dx dt < \infty \quad (2.9)$$

and that there exists  $f : [0, T] \rightarrow H$  satisfying (2.6) such that  $\int_0^T |f(t)| dt < \infty$ . Then there exists the mild solution of (2.7).

*Proof.* One has  $|B(t)|_2^2 = \sum_{k=1}^{\infty} |B(t)e_k|^2$ , and  $B(t)e_k = \sigma^k(t, \cdot)$ , hence

$$|B(t)e_k|^2 = \int_0^{\infty} |\sigma_x^k(t, x)|^2 \alpha(x) dx.$$

Therefore (2.9) implies  $\int_0^T |B(t)|_2^2 dt < \infty$ , and the result follows by proposition 3.  $\square$

Much simpler conditions can be stated if the driving noise  $Y_0$  is finite dimensional.

**Proposition 6.** *Assume that  $Y_0$  is a  $\mathbb{R}^n$ -valued Lévy process such that  $\int_{\mathbb{R}^n} e^{\langle \xi, z \rangle} m_0(d\xi) < \infty$  for all  $z \in B_r$ , for some  $r > 0$ . If  $\int_0^x \sigma(t, y) dy \in B_r$  for all  $(t, x) \in [0, T] \times \mathbb{R}_+$ , and*

$$\int_0^T \left( \int_0^{\infty} |\sigma_x^k(t, x)|^2 \alpha(x) dx \right)^2 dt < \infty \quad (2.10)$$

for all  $k = 1, \dots, n$ , then (2.6) reduces to

$$f(t, x) = - \left\langle \sigma(t, x), D\psi \left( - \int_0^x \sigma(t, y) dy \right) \right\rangle, \quad (t, x) \in [0, T] \times \mathbb{R}_+, \quad (2.11)$$

and (2.7) admits a mild solution.

*Proof.* It is enough to check that the hypotheses of proposition 3 are satisfied. In particular, similarly as before, one has, using Jensen's inequality,

$$\begin{aligned} \int_0^T |B(t)|_2^2 dt &= \int_0^T \sum_{k=1}^n \int_0^\infty |\sigma_x^k(t, x)|^2 \alpha(x) dx dt \\ &= \sum_{k=1}^n \int_0^T |\sigma^k(t, \cdot)|_H^2 dt \leq \sum_{k=1}^n \left( T \int_0^T |\sigma^k(t, \cdot)|_H^4 dt \right)^{1/2} < \infty \end{aligned}$$

where the last inequality is immediate by (2.10). Moreover, in (2.11) the quantity  $D\psi(x)$  is well defined for  $|x| \leq r$  because  $\psi \in C^\infty(B_r)$ . This implies that there exists a positive constant  $N$  such that  $|\psi_{x^i}(z)| < N$ ,  $|\psi_{x^i x^j}(z)| < N$  for all  $z \in B_r$ . We have

$$\begin{aligned} f_x(t, x) &= - \left\langle \sigma_x(t, x), D\psi \left( - \int_0^x \sigma(t, y) dy \right) \right\rangle \\ &\quad + \left\langle D^2\psi \left( - \int_0^x \sigma(t, y) dy \right) \sigma(t, x), \sigma(t, x) \right\rangle, \end{aligned}$$

hence  $\int_0^T |f(t)|_H dt < \infty$  if  $\int_0^T \int_0^\infty [\sigma^i(t, x) \sigma^j(t, x)]^2 \alpha(x) dx dt < \infty$  for all  $i, j \leq n$ . The condition  $\int_0^x \sigma(t, y) dy < r$  for all  $x \geq 0$  implies that  $\lim_{y \rightarrow \infty} |\sigma(t, y)| = 0$  for all  $t \in [0, T]$ , hence, using Cauchy-Schwarz' inequality,

$$\begin{aligned} &\int_0^T \int_0^\infty [\sigma^i(t, x) \sigma^j(t, x)]^2 \alpha(x) dx dt \\ &\leq \left( \int_0^T \int_0^\infty \sigma^i(t, x)^4 \alpha(x) dx dt \right)^{1/2} \left( \int_0^T \int_0^\infty \sigma^j(t, x)^4 \alpha(x) dx dt \right)^{1/2} \\ &\leq N(\alpha) \left( \int_0^T |\sigma^i(t, \cdot)|_H^4 dt \right)^{1/2} \left( \int_0^T |\sigma^j(t, \cdot)|_H^4 dt \right)^{1/2} \end{aligned}$$

where the second inequality follows by (5.8) in [10].  $\square$

An analogous expression could be obtained for a general Hilbert space valued noise  $Y$ , if one can guarantee that  $\psi$  is Fréchet differentiable. In the next proposition we shall identify the Fréchet derivative  $D\psi(x) \in \mathcal{L}(K, \mathbb{R})$  with its Riesz representative vector in  $K$ .

**Proposition 7.** Assume that  $\int_K e^{\langle \xi, z \rangle} m_0(d\xi) < \infty$  for all  $z \in B_r$ ,  $\psi \in C_b^2(B_r)$ , for some  $r > 0$ , and

$$\int_0^T \left( \sum_{k=1}^\infty \int_0^\infty |\sigma_x^k(t, x)|^2 \alpha(x) dx \right)^2 dt < \infty \quad (2.12)$$

If  $\int_0^x \sigma(t, y) dy \in B_r$  for all  $(t, x) \in [0, T] \times \mathbb{R}_+$ , then (2.6) reduces to (2.11), where  $\langle \cdot, \cdot \rangle$  is the inner product of  $K$ , and equation (2.7) admits a mild solution.

*Proof.* As seen before, we have

$$|B(t)|_2^2 = \sum_{k=1}^\infty \int_0^\infty |\sigma_x^k(t, x)|^2 \alpha(x) dx,$$

hence, by (2.12),  $\int_0^T |B(t)|_2^2 dt \leq (T \int_0^T |B(t)|_2^4 dt)^{1/2} < \infty$ . Let us prove that  $\int_0^T |f(t)| dt < \infty$ . The same expression for  $f_x(t, x)$  as in the proof of the previous proposition holds, mutatis mutandis. Therefore, in view of (2.12) and  $\psi \in C_b^2(B_r)$ , it is enough to show that  $\int_0^T \int_0^\infty |\sigma(t, x)|^4 \alpha(x) dx dt < \infty$ . Let  $(e_k)$  be a basis of  $K$ , and set  $\sigma_n(t, x) = \sum_{k=1}^n \sigma^k(t, x) e_k$ . Let  $\phi_n^{(\varepsilon)}$  a smooth approximation of  $x \mapsto |x|$  in  $\mathbb{R}^n$  such that  $|D\phi_n^{(\varepsilon)}| \leq 1$ , then we have as follows by (5.8) in [10]

$$\begin{aligned} \int_0^\infty \phi_n^{(\varepsilon)}(\sigma_n(t, x))^4 \alpha(x) dx &\leq N |\phi_n^{(\varepsilon)}(\sigma_n(t, \cdot))|_H^4 \\ &= N \left( \int_0^\infty |D_x \phi_n^{(\varepsilon)}(\sigma_n(t, x))|^2 \alpha(x) dx \right)^2 \\ &\leq N \left( \int_0^\infty |D_x \sigma_n(t, x)|^2 \alpha(x) dx \right)^2 \\ &\leq N \left( \sum_{k=1}^\infty \int_0^\infty |\sigma_x^k(t, x)|^2 \alpha(x) dx \right)^2 \end{aligned}$$

with  $N = N(\alpha)$ , thus by (2.12)

$$\int_0^T \int_0^\infty \phi_n^{(\varepsilon)}(\sigma_n(t, x))^4 \alpha(x) dx dt \leq N \int_0^T \left( \sum_{k=1}^\infty \int_0^\infty |\sigma_x^k(t, x)|^2 \alpha(x) dx \right)^2 dt < \infty. \quad (2.13)$$

Since the bound in (2.13) does not depend on  $\varepsilon$  nor on  $n$ , passing to the limit as  $\varepsilon \rightarrow 0$  we get  $\int_0^T \int_0^\infty |\sigma_n(t, x)|^4 \alpha(x) dx < \infty$ , and letting  $n$  tend to infinity we finally get  $\int_0^T \int_0^\infty |\sigma(t, x)|^4 \alpha(x) dx < \infty$ .  $\square$

### 3 Invariant measures and asymptotic behavior

In this section we assume  $\sigma(t, \cdot) \equiv \sigma(\cdot)$ , thus also  $B(t) \equiv B$ ,  $f(t) \equiv f$ . In view of the no-arbitrage considerations in the previous section, we also assume that  $m_0$  admits exponential moments, and  $B \in \mathcal{L}(K, H_0)$ , where  $H_0 := \{g \in H : g(\infty) = 0\}$ , hence also  $f \in H_0$ . In order for the following results to hold, it is not necessary to assume that  $f$  is such that no-arbitrage is verified, even though this is of course the situation we are interested in.

Let us rewrite (2.7), for convenience of notation, in the more compact form

$$du(t) = Au(t) dt + dY(t), \quad (3.1)$$

where  $Y(t) := ft + BY_0(t)$ . Then one can easily prove that  $Y$  is a  $H$ -valued Lévy process with triplet  $[b, R, m]$ , where

$$\begin{aligned} b &= f + Bb_0 + \int_K B\xi(\chi_1(B\xi) - \chi_1(\xi)) m_0(d\xi) \\ R &= BR_0B^* \\ m(d\xi) &= m_0(B^{-1}d\xi) \end{aligned}$$

The following proposition gives a simple sufficient condition for the existence and uniqueness of an invariant measure for an HJM model with deterministic volatility and Hilbert space valued Lévy noise. The only real requirement is that the state space  $L_{2,\alpha}^1$  is chosen with an exponentially growing weight  $\alpha$ .

**Proposition 8.** Assume that  $\alpha(x) = e^{\alpha x}$ ,  $\alpha > 0$ , and the forward curve at time zero is deterministic. Then there exists a unique invariant measure for (3.1) to which the law of  $u(t)$  weakly converges as  $t \rightarrow \infty$ .

*Proof.* Writing equation (3.1) in mild form, recalling that the range of  $B$  is contained in  $H_0$ , one recognizes that  $u(t, \infty) = u_0(\infty)$  for all  $t \geq 0$  (“long rates never fall”). Considering the isomorphism  $H = H_0 \oplus \mathbb{R}$ , (3.1) is equivalent to the system

$$\begin{cases} d\bar{u}(t) = A\bar{u}(t) dt + dY(t) \\ u(t, \infty) = \ell, \end{cases} \quad (3.2)$$

where  $\bar{u}$  is the projection of  $u$  on  $H_0$ ,  $A$  still denotes the restriction of  $A$  to  $H_0$ , and  $\ell \in \mathbb{R}$ . Let us show that  $e^{tA}$  is exponentially stable on  $H_0$ :

$$|e^{tA}\phi|_{H_0}^2 = \int_0^\infty \phi'(x+t)^2 \alpha(x) dx \leq e^{-\alpha t} \int_0^\infty \phi'(x)^2 \alpha(x) dx = e^{-\alpha t} |\phi|_{H_0}^2,$$

i.e.  $|e^{tA}| \leq e^{-t\alpha/2}$ . The obvious inequality  $x^2 \geq \log(1+x)$ ,  $x \geq 1$ , and the assumption  $\int_K |\xi|^2 m_0(d\xi) < \infty$  imply that

$$\begin{aligned} \int_{|\xi| \geq 1} \log(1+|\xi|) m(d\xi) &\leq \int_{|\xi| \geq 1} |\xi|^2 m_0(B^{-1}d\xi) \\ &\leq |B|^2 \int_K |\zeta|^2 m_0(d\zeta) < \infty. \end{aligned}$$

Therefore theorem 6.7 of [3] yields the existence of an invariant measure  $\bar{\mu}$  on  $H_0$  for the first equation of (3.2), hence  $\mu = \bar{\mu} \otimes \delta_\ell$  is an invariant measure for (3.1) on  $H$ . Since  $|e^{tA}| \leq e^{-t\alpha/2} \rightarrow 0$  as  $t \rightarrow \infty$  (i.e.  $e^{tA}$  is stable), proposition 6.1 of [3] (or theorem 3.1 of [12]) imply that  $\bar{\mu}$  is infinitely divisible with triplet  $[b_\infty, R_\infty, m_\infty]$ , where

$$b_\infty = \lim_{t \rightarrow \infty} \left[ \int_0^t e^{sA} b ds + \int_0^t \int_{H_0} e^{sA} \xi (\chi_1(e^{sA}\xi) - \chi_1(\xi)) m(d\xi) ds \right] \quad (3.3)$$

$$R_\infty = \int_0^\infty e^{sA} R e^{sA*} ds \quad (3.4)$$

$$m_\infty(d\xi) = \int_0^\infty m((e^{sA})^{-1}d\xi) ds, \quad m_\infty(\{0\}) = 0, \quad (3.5)$$

are all well defined thanks to the stability properties of  $e^{tA}$ . In particular  $\bar{\mu}$  is unique. Finally, by proposition 6.1 of [3] we have that  $\bar{\mu}$  coincides with the law of the random variable  $\int_0^\infty e^{sA} dY(s)$ , and lemma 3.1 of [3] allows to conclude that the law of  $\bar{u}(t)$  weakly converges to  $\bar{\mu}$ .  $\square$

*Remark 9.* The decomposition  $H = H_0 \oplus \mathbb{R}$  was already used in [29], but essentially the same “trick” already appeared, perhaps less explicitly, in [30]. The fast growth at infinity of the weight  $\alpha$  was needed in [29] as well, where it is assumed that  $\alpha := \inf_{x \geq 0} \alpha'(x)/\alpha(x) > 0$ . In fact, Gronwall’s lemma immediately yields that this condition implies  $\alpha(x) \geq \alpha(0)e^{\alpha x}$ .

The choice of the weight function  $\alpha$ , as just seen, determines the stability properties of the semigroup  $e^{tA}$ . For a generic choice of  $\alpha$  we cannot guarantee exponential stability of  $e^{tA}$  in  $H_0$ , but we still have stability, in the sense that  $|e^{tA}g|_{H_0} \rightarrow 0$  as  $t \rightarrow \infty$  for any  $g \in H_0$ . However, in order to obtain existence of an invariant measure for (3.1), the conditions to verify become quite difficult, in general. In particular the following characterization holds, the proof of which follows [3] or [12].

**Proposition 10.** *Assume that the forward curve at time zero is deterministic. The following conditions are sufficient and necessary for the existence of a (unique) invariant measure  $\mu$  for (3.1):*

- (i)  $\sup_{t \geq 0} \text{Tr} \int_0^t e^{sA} B R_0 B^* e^{sA^*} ds;$
- (ii)  $\int_0^\infty \int_{H_0} (|e^{tA}x|^2 \wedge 1) m(dx) dt < \infty;$
- (iii) *the limit in (3.3) exists.*

Moreover,  $\mu = \bar{\mu} \otimes \delta_\ell$ , where  $\bar{\mu}$  is infinitely divisible with triplet  $[b_\infty, R_\infty, m_\infty]$  given by (3.3)-(3.5). Finally, the law of  $u(t)$  weakly converges to  $\bar{\mu}$  at  $t \rightarrow \infty$ .

*Proof.* The semigroup  $e^{tA}$  is stable on  $H_0$  because

$$|e^{tA}g|_{H_0}^2 = \int_t^\infty g'(x)^2 \alpha(x-t) dx \leq \int_t^\infty g'(x)^2 \alpha(x) dx \xrightarrow{t \rightarrow \infty} 0, \quad (3.6)$$

where the inequality follows by monotonicity of  $\alpha$  and the limit is zero because the integrand is in  $L_1(\mathbb{R}_+)$ . Therefore theorem 6.4 of [3] (or theorem 3.1 of [12]) implies that the infinitely divisible measure  $\bar{\mu}$  on  $H_0$  with triplet  $[b_\infty, R_\infty, m_\infty]$  is invariant for the first equation of (3.2). The proof is then completed exactly as in the previous proposition.  $\square$

**Corollary 11.** *Assume that the forward curve at time zero is deterministic and that there exists a function  $\phi \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$  such that  $|e^{tA}x|_{H_0} \leq \phi(t)|x|_{H_0}$ . Then  $\mu$  as defined in the previous proposition is the unique invariant measure of (3.1) and it is ergodic.*

*Proof.* Since  $e^{tA}$  is a stable semigroup on  $H_0$ , it is enough to verify hypotheses (i)–(iii) of the last proposition. We have

$$\begin{aligned} \int_0^\infty \int_{H_0} (|e^{tA}x|^2 \wedge 1) m(dx) dt &\leq \int_0^\infty \int_{H_0} \phi(t)^2 |x|^2 m(dx) dt \\ &\leq \int_0^\infty \phi(t)^2 dt |B|^2 \int_K |x|^2 m_0(dx) < \infty, \end{aligned}$$

because  $\int_K |x|^2 m_0(dx) < \infty$ . Since  $m_0$  admits exponential moments, then  $\int_K |x| m_0(dx) < \infty$ , and

$$\int_0^\infty \int_{H_0} |e^{tA}x(\chi_1(e^{tA}x) - \chi_1(x))| m(dx) dt \leq 2 \int_0^\infty \phi(t) dt |B| \int_K |x| m_0(dx) < \infty.$$

Let us now prove that  $\lim_{t \rightarrow \infty} \int_0^t e^{sA} b \, ds$  exists in  $H_0$ : we have

$$\bar{b}(x) := \lim_{t \rightarrow \infty} \left[ \int_0^t e^{sA} b \, ds \right](x) = \int_x^\infty b(s) \, ds, \quad x \geq 0,$$

thus  $\bar{b}'(x) = -b(x)$  and

$$\int_0^\infty |e^{sA} b| \, ds \leq |b|_{H_0} \int_0^\infty \phi(s) \, ds < \infty,$$

i.e.  $b_\infty \in H_0$ . Similarly we have

$$\text{Tr } R_\infty = \sup_{t \geq 0} \int_0^t e^{sA} B R_0 B^* e^{sA^*} \, ds \leq |B|^2 \text{Tr } R_0 \int_0^\infty \phi(s)^2 \, ds < \infty,$$

i.e.  $R_\infty$  is well defined.  $\square$

*Remark 12.* The semigroup  $e^{sA}$  is a contraction semigroup for any choice of  $\alpha$ , e.g. by (3.6), hence if  $\phi$  as in the previous corollary exists, then one can always choose  $|\phi(t)| \leq 1$ , thus  $\phi \in L_1(\mathbb{R}_+)$  also implies  $\phi \in L_2(\mathbb{R}_+)$ . A possible choice of  $\phi$  (although very rough) could be

$$\phi(t) = \sup_{x \geq 0} \left( \frac{\alpha(x)}{\alpha(x+t)} \right)^{1/2},$$

provided that  $\int_0^\infty \sup_{x \geq 0} (\alpha(x)/\alpha(x+t))^{1/2} \, dt < \infty$ .

We have seen in the previous proposition that one of the necessary and sufficient conditions for the existence of an invariant measure is that  $b_\infty$  exists. In particular, if  $m_0$  is symmetric, the problem reduces to proving that  $\bar{b} := \int_0^\infty e^{sA} b \, ds$  is a well-defined element of  $H_0$ . In fact, if  $m_0$  is symmetric then  $m$  is symmetric as well, and the second summand on the right hand side of (3.3) is zero for all  $t \geq 0$ . It is thus natural to look for conditions on  $\alpha$  such that the norm of  $\bar{b}$  can be bounded in terms of the norm of  $b$ . This is indeed possible, and one can give a sharp condition, namely (3.7) below is necessary and sufficient for  $|\bar{b}| \leq N|b|$  to hold.

**Proposition 13.** *Assume that*

$$\sup_{x \geq 0} \int_0^x \alpha(y) \, dy \int_x^\infty \frac{1}{\alpha(y)} \, dy < \infty. \quad (3.7)$$

*Then  $\int_0^\infty e^{sA} b \, ds$  exists in  $H_0$ .*

*Proof.* As in the proof of corollary 11, we only need to prove that  $b^2 \alpha \in L_1(\mathbb{R}_+)$ . Let  $\nu$  be a nonnegative Borel measure on  $\mathbb{R}_+$ . By a result of Muckenhoupt [22], we have that the following weighted Hardy inequality holds for all measurable functions  $f$

$$\int_0^\infty \left| \int_0^x f(y) \, dy \right|^2 \nu(dx) \leq N \int_0^\infty f(x)^2 \nu(dx),$$

where  $N$  is a positive constant that depends only on  $\alpha$ , if and only if

$$\sup_{r \geq 0} [\nu([r, +\infty))^{1/2} \left[ \int_0^r \left( \frac{d\nu}{dx} \right)^{-1} dx \right]^{1/2}] < \infty.$$



By the change of variable  $x \mapsto x^{-1}$ , choosing  $\nu(dx) = \alpha(x) dx$ , we obtain that (3.7) is necessary and sufficient for

$$\int_0^\infty \left| \int_x^\infty b'(y) dy \right|^2 \alpha(x) dx \leq N \int_0^\infty b'(x)^2 \alpha(x) dx.$$

Since  $b \in H_0$ , we have that  $|\int_x^\infty b'(y) dy| = |b(x)|$  and that the right-hand side of the previous inequality is finite.  $\square$

## 4 Conclusions

We have considered an equation of HJM type driven by a Lévy process taking values in a Hilbert space, obtaining sufficient conditions for the absence of arbitrage. Assuming that the volatility operator is deterministic, and using Musiela's parametrization, one obtains a stochastic evolution equation of Ornstein-Uhlenbeck type. We have discussed existence of mild solutions and existence, uniqueness, and ergodicity of invariant measures, generalizing previous work of Vargiolu [30], who considered the situation of a driving Brownian motion and used a different state space. The choice of the state space  $L_{2,\alpha}^1$  seems to be the standard by now, since its elements enjoy most desirable features for a forward curve. If the weight function  $\alpha$  grows exponentially at infinity, the HJM dynamics admits a unique invariant measure. A similar conclusion was obtained by Tehranchi in [29], where a Musiela equation with state-dependent volatility and Brownian noise was considered, and the results of [6] could be applied. It would be natural to consider also in the setting of Lévy noise a state-dependent volatility operator, but unfortunately there are comparatively very few results on evolution equations with jump noise, which need to be established first.

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