An empirical analysis of multivariate copula models
Fischer, Matthias; Köck, Christian; Schlüter, Stephan; Weigert, Florian

Postprint / Postprint
Zeitschriftenartikel / journal article

Zur Verfügung gestellt in Kooperation mit / provided in cooperation with:
www.peerproject.eu

Empfohlene Zitierung / Suggested Citation:

Nutzungsbedingungen:
Mit der Verwendung dieses Dokuments erkennen Sie die Nutzungsbedingungen an.

Terms of use:
This document is made available under the "PEER Licence Agreement ". For more Information regarding the PEER-project see: http://www.peerproject.eu This document is solely intended for your personal, non-commercial use. All of the copies of this documents must retain all copyright information and other information regarding legal protection. You are not allowed to alter this document in any way, to copy it for public or commercial purposes, to exhibit the document in public, to perform, distribute or otherwise use the document in public.
By using this particular document, you accept the above-stated conditions of use.

Diesse Version ist zitierbar unter / This version is citable under:
https://nbn-resolving.org/urn:nbn:de:0168-ssoar-221383
# AN EMPIRICAL ANALYSIS OF MULTIVARIATE COPULA MODELS

<table>
<thead>
<tr>
<th>Journal:</th>
<th>Quantitative Finance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID:</td>
<td>RQUF-2007-0042.R1</td>
</tr>
<tr>
<td>Manuscript Category:</td>
<td>Research Paper</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>19-Nov-2007</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Fischer, Matthias; HSU Hamburg, Computational Statistics; University Erlangen-Nürnberg, Statistics and Econometrics Köck, Christian; University Erlangen-Nürnberg, Statistics and Econometrics Schlüter, Stephan; University Erlangen-Nürnberg Weigert, Florian; University Erlangen-Nürnberg</td>
</tr>
<tr>
<td>Keywords:</td>
<td>Applied Econometrics, Applied Finance, Copulas, Financial Econometrics</td>
</tr>
<tr>
<td>JEL Code:</td>
<td>C13 - Estimation &lt; C1 - Econometric and Statistical Methods: General &lt; C - Mathematical and Quantitative Methods, C16 - Specific Distributions &lt; C1 - Econometric and Statistical Methods: General &lt; C - Mathematical and Quantitative Methods</td>
</tr>
</tbody>
</table>

E-mail: quant@tandf.co.uk  URL://http.manuscriptcentral.com/tandf/rquf
AN EMPIRICAL ANALYSIS OF MULTIVARIATE COPULA MODELS

MATTHIAS FISCHER†, CHRISTIAN KÖCK‡, STEPHAN SCHLÜTER & FLORIAN WEIGERT

† Department of Computational Statistics
Helmut-Schmidt University of Hamburg, Germany
Email: mfischer@hsu-hh.de

‡ Department of Statistics and Econometrics
University of Erlangen-Nürnberg, Germany
Email: Christian.Koeck@wiso.uni-erlangen.de

This research was kindly supported by the Hans Frisch-Stiftung.

SUMMARY
Since the pioneering work of Embrechts and co-authors in 1999, copula models have enjoyed steadily increasing popularity in finance. Whereas copulas are well-studied in the bivariate case, the higher-dimensional case still offers several open issues and it is far from clear how to construct copulas which sufficiently capture the characteristics of financial returns. For this reason, elliptical copulas (i.e. Gaussian and Student-t copula) still dominate both empirical and practical applications. On the other hand, several attractive construction schemes have appeared in the recent literature promising flexible but still manageable dependence models. The aim of this work is to empirically investigate whether these models are really capable of outperforming its benchmark, i.e. the Student-t copula and, in addition, to compare the fit of these different copula classes among themselves.

Keywords and phrases: KS-copula; Hierarchical Archimedian; Product copulas; Pair-copula decomposition

1 Introduction
The increasing linkages between countries, markets and companies require an accurate and realistic modelling of the underlying dependence structure. This applies to financial markets and, in particular, to the financial assets traded there-on. For a long time both practitioners and theorists have relied on the multivariate normal (Gaussian) distribution as statistical foundation, seemingly ignoring that this model assigns too little probability mass to extremal events. In order to overcome this drawback but still maintain many of the attractive properties, elliptical distributions (e.g. multivariate Student-t or multivariate generalized hyperbolic distribution) occasionally have found their way into financial literature. Though being able to model heavy tails, elliptical distributions fail to capture asymmetric dependence structures. The copula concept, in contrast, which originally dates back to Sklar (1959) but was made popular in finance through the pioneering work of Embrechts and co-authors (1999), provides a flexible tool to capture different patterns of dependence. Within
this work we assume that the reader is already familiar with the notion of copulas. Otherwise, we refer to Nelsen (2006) or Joe (1997). Whereas copulas are well-studied in the bivariate case, construction schemes for higher dimensional copulas are not. Recently, several publications on high-dimensional copulas have appeared (e.g. Morillas, 2005, Palmitesta & Provasi, 2005, Savu & Trede, 2006, Liebscher, 2006, Aas et al., 2006). Each of them claims to provide a flexible dependence model, but there is no comprehensive comparison among these approaches, as far as we know. In particular, no references are found to the Student-t copula (i.e. the copula associated to the multivariate Student-t distribution) which is known for its excellent fit to multivariate financial return data.

The outline of this work is as follows: Section 2 overviews and connects several recent construction schemes of multivariate copulas. A short digression on goodness-of-fit measures can be found in section 3. Section 4 is dedicated to the description of the underlying data sets, whereas the empirical results are summarized and discussed in section 5.

2 Constructing multivariate non-elliptical copulas

Among the classes of non-elliptical copulas, Archimedean copulas and its generalizations (section 2.1) enjoy great popularity. Beyond that, so-called pair-copula constructions are reviewed in section 2.2, where the joint distribution is decomposed into simple building blocks, so-called pair-copulas. Thirdly, we pick up the copulas associated to Koehler-Symanowski distributions in section 2.3 which have been successfully applied by Palmistesta & Provasi (2005) as models for financial returns. Finally, Liebscher’s (2006) recent proposal to generalize given d-copulas is reviewed in section 2.4.

2.1 Multivariate Archimedean copulas

2.1.1 Classical multivariate Archimedean copulas

Let \( \varphi : [0, 1] \to [0, \infty] \) be a continuous, strictly decreasing and convex function with \( \varphi(1) = 0 \), \( \varphi(0) \leq \infty \) and let \( \varphi^{-1} \) be the so-called pseudo-inverse of \( \varphi \) defined by

\[
\varphi^{-1}(t) = \begin{cases} 
\varphi^{-1}(t) & 0 \leq t \leq \varphi(0), \\
0 & \varphi(0) \leq t \leq \infty.
\end{cases}
\]

It can be shown (see, e.g. Nelsen, 2006) that

\[
C(u_1, u_2) = \varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)
\]

defines a class of bivariate copulas, the so-called Archimedean copulas. The function \( \varphi \) is called the (additive) generator of the copula. Furthermore, if \( \varphi(0) = \infty \) the pseudo-inverse describes an ordinary inverse function (i.e. \( \varphi^{-1} = \varphi^{-1} \)) and in this case \( \varphi \) is known as a strict generator.
Given a strict generator \( \varphi : [0, 1] \to [0, \infty] \), bivariate Archimedean copulas can be extended to the \( d \)-dimensional case. For every \( d \geq 2 \) the function \( C : [0, 1]^d \to [0, 1] \) defined as

\[
C(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_d))
\]

(2.1)
is a \( d \)-dimensional Archimedean copula if and only if \( \varphi^{-1} \) is completely monotonic on \( \mathbb{R}_+ \), i.e. if \( \varphi^{-1} \in L_\infty \) with

\[
L_m \equiv \left\{ \phi : \mathbb{R}_+ \to [0, 1] \bigg| \phi(0) = 1, \phi(\infty) = 0, (-1)^k \phi^{(k)}(t) \geq 0, k = 1, \ldots, m \right\}.
\]

The Gumbel copula is derived from the generator \( \varphi(t) = (-\ln t)^\theta, \theta \geq 1 \) and the Clayton copula is generated by \( \varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1), \theta > 0 \). For an overview of further Archimedean copulas and the properties of the aforementioned ones, we refer the reader to the monographs of Nelson (2006) and Joe (1997).

### 2.1.2 Non-exchangeable Archimedean copulas

In order to increase flexibility and to allow for non-exchangeable dependence structures, several generalizations have emerged in the recent literature: A simple one – the so-called fully nested Archimedean (FNA) copulas – can be found in Joe (1997, p. 89), Whelan (2004) and Savu & Trede (2006), and requires \( d - 1 \) generator functions \( \varphi_1, \ldots, \varphi_{d-1} \) with \( \varphi_1^{-1}, \ldots, \varphi_{d-1}^{-1} \in L_\infty \) and \( \varphi_i \circ \varphi_j^{-1} = \varphi_i(\varphi_j^{-1}(t)) \in L_\infty \) for

\[
L_d = \left\{ \phi : \mathbb{R}_+ \to \mathbb{R}_+ \bigg| \phi(0) = 0, \phi(\infty) = \infty, (-1)^k \phi^{(k)}(t) \geq 0, k = 1, \ldots, d \right\}.
\]

The structure of FNA \( d \)-copulas is rather simple: One first couples \( u_1 \) and \( u_2 \), then the copula of \( u_1 \) and \( u_3 \) with \( u_4 \) to a new copula which is coupled afterwards with \( u_4 \) and so on. Hence the FNA 4-copula is of the form

\[
C(\mathbf{u}) = \varphi_3^{-1} \left[ \varphi_3 \left( \varphi_2^{-1} \left[ \varphi_2 \left( \varphi_1^{-1} (u_1) + \varphi_1(u_2) \right) \right] + \varphi_2(u_3) \right) \right] + \varphi_3(u_4).
\]

(2.2)

Figure 1 illustrates one possible FNA copula for dimension \( d = 5 \).

Alternatively, mixing ordinary Archimedean and FNA copulas, partially nested Archimedean (PNA) copulas may be used. Again, for ease of notation, we focus on the 4-variate case

\[
C(\mathbf{u}) = \varphi^{-1} \left[ \varphi \left( \varphi_{12}^{-1} [\varphi_{12}(u_1) + \varphi_{12}(u_2)] \right) + \varphi \left( \varphi_{34}^{-1} [\varphi_{34}(u_3) + \varphi_{34}(u_4)] \right) \right].
\]

(2.3)

Note that \( \varphi, \varphi_{12}, \varphi_{34} \) are generators with \( \varphi^{-1}, \varphi_{12}^{-1}, \varphi_{34}^{-1} \in L_\infty \) and \( \varphi \circ \varphi_{12}^{-1}, \varphi \circ \varphi_{34}^{-1} \in L_\infty \). Obviously, one first couples the pairs \( u_1, u_2 \) and \( u_3, u_4 \) with distinct generators. The resulting copula pair is then coupled using a third generator \( \varphi \) (which in turn might be coupled with an additional variable \( u_5 \) using a fourth generator \( \psi \) for an extension to the 5-dimensional case). Another possible structure of a PND copula is illustrated in figure 1.
Figure 1: FNA copula (left) and PNA copula (right) for \( d = 5 \).

Thirdly, copula \( C \) from (2.3) is also an example of a so-called hierarchical Archimedean (HA) copula. Borrowing the notation of Savu & Trede (2006), the basic idea of this approach is to build a hierarchy of Archimedean copulas with \( L \) different levels, indexed by \( l = 1, \ldots, L \). At each level \( l \) there are \( n_l \) distinct objects, indexed by \( j = 1, \ldots, n_l \). In a first step (i.e. in level 1), the variables \( u_1, \ldots, u_d \) are grouped into \( n_1 \) ordinary multivariate Archimedean copulas

\[
C_{1,j}(u_{1,j}) = \varphi_{1,j}^{-1}\left(\sum_{u_{1,j}} \varphi_{1,j}(u_{1,j})\right), \quad j = 1, \ldots, n_1.
\]

with (possibly different) generators \( \varphi_{1,j} \) and where \( u_{1,j} \) denotes the set of elements of \( u_1, \ldots, u_d \) belonging to \( C_{1,j} \). All copulas of the first level are again grouped into copulas at level \( l = 2 \). These copulas \( C_{2,j} \) with generator function \( \varphi_{2,j} \), \( j = 1, \ldots, n_2 \) are generalized Archimedean copulas, whose dependence structure is only of partial exchangeability and consists of copulas from the previous level (as elements), denoted by

\[
C_{2,j}(C_{2,j}) = \varphi_{2,j}^{-1}\left(\sum_{C_{2,j}} \varphi_{2,j}(C_{2,j})\right),
\]

where \( C_{2,j} \) represents the set of all copulas from level \( l = 1 \) entering copula \( C_{2,j} \). This procedure continues until only a single hierarchical Archimedean copula \( C_{L,1} \) is achieved at level \( L \). In order to ensure that \( C_{L,1} \) is a proper copula, we have to proclaim that \( \varphi_{l,j}^{-1} \in \mathcal{L}_\infty \) for \( l = 1, \ldots, L \) and \( j = 1, \ldots, n_l \), and that \( \varphi_{l+1,i} \circ \varphi_{l,j}^{-1} \in \mathcal{L}_\infty^* \) for all \( l = 1, \ldots, L \) and \( j = 1, \ldots, n_l, i = 1, \ldots, n_{l+1} \) such that \( C_{l,j} \in C_{l+1,i} \). Moreover, a hierarchy is established if the number of copulas decreases at each level, if the top level contains only a single object and if at each level the dimensions of the copulas add up to \( d \). Figure 2 displays the possible construction of a 5-dimensional HA-copula.

Savu & Trede (2006) also derive the HA-copula density

\[
\frac{\partial^d C_{L,1}}{\partial u_1 \ldots \partial u_d} = \sum_{\partial C_{L-1,1}^<} \cdots \sum_{\partial C_{L-1,n_{L-1}}^<} \frac{\partial^{d-i} C_{L,1}}{\partial C_{L-1,1} \cdots \partial C_{L-1,n_{L-1}}} \prod_{r=1}^{n_{L-1}} \sum_{v_1=v_{1,r}, \ldots, v_r} \frac{\partial^{|v_1|} C_{L-1,r}}{\partial v_1} \cdots \frac{\partial^{|v_r|} C_{L-1,r}}{\partial v_r}.
\]

E-mail: quant@tandf.co.uk URL://http.manuscriptcentral.com/tandf/rquf
In this section we focus on methods recently proposed by Morillas (2005) and Liebscher (2006). Both approaches are based on a second functional representation of Archimedean copulas via so called multiplicative generators (see Nelsen, 2006). Setting \( \vartheta(t) \equiv \exp(-\varphi(t)) \) and \( \vartheta^{-1}(t) \equiv \varphi^{-1}(-\ln t) \), equation (2.1) can be rewritten as

\[
C(u_1, \ldots, u_d) = \vartheta^{-1}\left(\vartheta(u_1) \cdot \vartheta(u_2) \cdot \ldots \cdot \vartheta(u_d)\right). \tag{2.4}
\]

The function \( \vartheta \) is called multiplicative generator of \( C \). Due to the relationship between \( \varphi \) and \( \vartheta \), the function \( \vartheta : [0, 1] \to [0, 1] \) is continuous, strictly increasing and concave with \( \vartheta(1) = 1 \) and \( \vartheta^{-1}(t) = 0 \) if \( 0 \leq t \leq \vartheta(0) \) and \( \vartheta^{-1}(t) = \vartheta^{-1}(t) \) if \( \vartheta(0) \leq t \leq 1 \).

Equation (2.4) can also be expressed using the independence copula \( C^\perp(u) = \prod_{i=1}^d u_i \):

\[
C(u_1, \ldots, u_d) = \vartheta^{-1}\left(C^\perp(\vartheta(u_1), \ldots, \vartheta(u_d))\right).
\]

Morillas (2005) substitutes \( C^\perp \) by an arbitrary \( d \)-copula \( C \) in order to obtain

\[
C_{\vartheta}(u_1, \ldots, u_d) = \vartheta^{-1}\left(C(\vartheta(u_1), \vartheta(u_2), \ldots, \vartheta(u_d))\right) \tag{2.5}
\]

and proves that \( C_{\vartheta} \) is a \( d \)-copula if \( \vartheta^{-1} \) is absolutely monotonic of order \( d \) on \( [0, 1] \), i.e. if \( \vartheta^{-1}(t) \) satisfies \( (\vartheta^{-1})^{(k)}(t) = \frac{d^k \vartheta^{-1}(t)}{dt^k} \geq 0 \) for \( k = 1, 2, \ldots, d \) and \( t \in (0, 1) \).

Examples of generator functions are stated in Morillas (2005). Notice that not every generator given there is absolutely monotonic for arbitrary \( d > 1 \): As one can easily verify, the generator \( \vartheta(t) = t^r / (2 - t^r) \), \( r \in (0, 1/3] \) (see table 1, no. 9 in Morillas, 2005) has no absolutely monotonic pseudo-inverse of order \( d \geq 3 \), because the third derivative of \( \vartheta^{-1} \) becomes negative. Hence this generator is suitable only for a construction of generalized copulas from...
bivariate copulas. Concerning the basic properties of such Morillas copulas we refer to Morillas (2005).

Another way of generalizing Archimedean copulas is the method proposed by Liebscher (2006) who introduces the following copula representation

\[
C(u_1, \ldots, u_d) = \Psi \left( \frac{1}{m} \sum_{j=1}^{m} \psi_{j1}(u_1) \cdot \psi_{j2}(u_2) \cdot \ldots \cdot \psi_{jd}(u_d) \right), \quad (2.6)
\]

where \( \Psi \) and \( \psi_{jk} : [0, 1] \to [0, 1] \) are functions satisfying the following conditions: Firstly, it is assumed that \( \Psi^{(d)} \) exists with \( \Psi^{(k)}(u) \geq 0 \) for \( k = 1, 2, \ldots, d \) and \( u \in [0, 1] \), and that \( \Psi(0) = 0 \). Secondly, \( \psi_{jk} \) is assumed to be differentiable and monotone increasing with \( \psi_{jk}(0) = 0 \) and \( \psi_{jk}(1) = 1 \) for all \( k, j \). Thirdly, Liebscher’s construction requires that

\[
\Psi \left( \frac{1}{m} \sum_{j=1}^{m} \psi_{jk}(v) \right) = v \quad \text{for} \quad k = 1, 2, \ldots, d \quad \text{and} \quad v \in [0, 1].
\]

The three conditions guarantee that \( C \) defined in (2.6) is actual a copula.

It is easily seen that the approaches of Morillas (2005) and Liebscher (2006) coincide for \( m = 1, \vartheta^{[-1]} = \Psi \) in (2.6) and \( C_\vartheta = C_{\bar{\vartheta}} \) in (2.5).

Liebscher (2006) also states a general method for deriving appropriate functions \( \psi_{jk} \). Let \( h_{jk} : [0, 1] \to [0, 1], j = 1, \ldots, m, k = 1, \ldots, d \) be a differentiable and bijective function such that \( h'_{jk}(u) > 0 \) for \( u \in (0, 1) \), \( h_{jk}(0) = 0 \), \( h_{jk}(1) = 1 \) and \( m \cdot u = \sum_{j=1}^{m} h_{jk}(u), u \in [0, 1], k = 1, \ldots, d \). Let \( \psi = \Psi^{-1} \) be the differentiable inverse function of \( \Psi \). An appropriate choice is setting \( \psi_{jk}(u) = h_{jk}(\psi(u)) \), since \( \psi'_{jk}(u) = h'_{jk}(\psi(u)) \cdot \psi'(u) > 0 \) for \( j = 1, \ldots, m \) and \( u \in [0, 1] \).

Considering \( m = 2 \), define

\[
h_{1k}(u) \equiv u^\delta, \quad h_{2k}(u) \equiv 2u - u^\delta \quad \text{with} \quad \delta_k \in [1, 2]. \quad (2.7)
\]

Choosing further

\[
\Psi(t) = -\frac{1}{\theta} \ln(1 - (1 - e^{-\theta})t), \quad \text{and} \quad \psi(u) = \frac{1 - e^{-\theta u}}{1 - e^{-\theta}}, \quad \theta > 0.
\]

and defining \( \psi_{jk}(u) = h_{jk}(\psi(u)) \) a generalized Frank copula (GMLF) is obtained. Setting \( \delta_k = 1 \) for all but one \( k, k = 1, \ldots, d \), it is easily verified that the GMLF copula reduces to the common Frank copula. Setting \( m = 2 \) and \( h_{jk} \) as in (2.7) but now choosing (see table 2, no. 2, p. 8 in Liebscher (2006))

\[
\Psi(t) = \frac{(\delta - t)^{-\theta} - \delta^{-\theta}}{(\delta - 1)^{-\theta} - \delta^{-\theta}}, \quad \text{and} \quad \psi(u) = \delta - \left( \frac{\delta^{-\theta}(1 - u) + u(\delta - 1)^{-\theta}}{1 - \theta} \right)^{-\frac{1}{\theta}}, \quad \theta > 0, \delta > 1
\]
a copula is obtained, which will be termed as GML2 copula henceforth.

In the field of insurance pricing the function $\psi_{jk}$ is known as a distortion function (for a
definition see Freez & Valdez, 1998) and the methods proposed by e.g. Freez & Valdez
(1998) or Wang (1998) appear as special cases in (2.6). The same holds for the approach
given by Morillas (2005) where the function $\vartheta$ also satisfies the requirements of a distortion
function.

2.2 Pair-copula decompositions

2.2.1 Pair-copula decomposition: The general case

One way of calculating a multivariate density is by decomposing it into a product of marginal
densities and conditional densities. The latter can be replaced stepwise by so-called pair-
copulas. Again, let $X = (X_1, \ldots, X_d)'$ have the joint density function

$$f(x_1, \ldots, x_d) = f(x_d) \cdot f(x_1|x_d) \cdot f(x_{d-1}|x_1, x_d) \cdots f(x_1|x_2, \ldots, x_d)$$

which is unique up to a relabelling of the variables. Because of

$$f(x_1, \ldots, x_d) = c_{12 \ldots d}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d),$$

with $c_{12 \ldots d}(\cdot)$ being the $d$-variate copula density, $f(x_d|x_{d-1})$, e.g., may also be expressed
by $c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1)$, where $c_{12}(\cdot, \cdot)$ is called pair-copula density for the respecting
transformed variables. Similarly, $f(x_{d-2}|x_{d-1}, x_d)$, can be decomposed into

$$c_{(d-2)d|d-1}(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1})) \cdot f(x_{d-2}|x_{d-1}).$$

Using $f(x_{d-2}|x_{d-1}) = c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2})$ results in

$$f(x_{d-2}|x_{d-1}, x_d) = c_{(d-2)d|d-1}(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1}))
\cdot c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2})$$

which is not unique anymore, because one may also condition on $x_d$ instead of $x_{d-1}$. This
leads to a different decomposition. The general formula reads as

$$f(x|v) = c_{xv_j|v_{-j}}(F(x|v_{-j}, F(v_j|v_{-j})) \cdot f(x|v_{-j})$$

for a $d$-dimensional vector $v$ with components $v_j$. The vector $v_{-j}$ denotes $v$ excluding the
component $v_j$. For methods and formulas to calculate $F(x|v)$ we refer to Joe (1996).

As seen above, every (conditional) $d$-dimensional density can be split up into a pair-copula
and a $(d-1)$-dimensional (conditional) density. For $d > 2$ you can iteratively repeat this
splitting for the $(d-1)$-dimensional conditional density. Eventually, a product of univariate
densities and pair-copulas is obtained. As shown in the trivariate case, this decomposition
is not unique but there are various ways of doing this.
In order to sort the different decomposition constructs, so-called regular vines (see Bedford and Cooke, 2001 and 2002) are defined. Vines are graphical models that present complete decomposition schemes. Following Aas et al. (2006) we choose the structure of the D-vine, since there is no dominating variable. The joint density $f(x_1, \ldots, x_d)$ can be expressed as

$$
\prod_{k=1}^{d} f(x_k) \prod_{j=1}^{d-j} c_{i,i+j|i+1,\ldots,i+j-1}(F(x_i|x_{i+1}, \ldots, x_{i+j-1}), F(x_{i+j}|x_{i+1}, \ldots, x_{i+j-1})).
$$

The decomposition of a four-dimensional density according to the D-vine scheme is

$$
f(x_1, x_2, x_3, x_4) = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4) \cdot c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \cdot c_{34}(F(x_3), F(x_4))
$$

$$
\cdot c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3))
$$

$$
\cdot c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3)).
$$

(2.10)

### 2.2.2 Pair-copula decomposition of a copula

Originally, the pair-copula decomposition (PCD) decomposes the common density $f$ of $d$ random variables. Of course, one may also apply the pair-copula decomposition to the underlying copula density $c$, as we will show in this subsection. To simplify notation, we restrict ourselves to $d = 4$ and the D-vine decomposition. As an immediate consequence of Sklar’s (1959) theorem,

$$
c(F(x_1), F(x_2), F(x_3), F(x_4)) = \frac{f(x_1, x_2, x_3, x_4)}{f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4)}.
$$

Substituting the common density by its PCD given in (2.10),

$$
c(F(x_1), F(x_2), F(x_3), F(x_4)) = c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \cdot c_{34}(F(x_3), F(x_4))
$$

$$
\cdot c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3))
$$

$$
\cdot c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3))
$$

with $c_{ij}(\cdot, \cdot)$ being a pair-copula density and its indices $i, j$ refer to $x_i$ and $x_j$. According to Joe (1996),

$$
F(x|v) = \frac{\partial C_{x,v_j|v_{-j}}(F(x|v_{-j}), F(v_j|v_{-j}))}{\partial F(v_j|v_{-j})}
$$

with $v_{-j}$ being the vector $v$ except the element $v_j$. In the univariate case (i.e., $v = v$),

$$
F(x|v) = \frac{\partial C_{x|v}(F_X(x), F_V(v))}{\partial F_V(v)} \equiv h(x, v, \theta),
$$

where $\theta$ being the parameter vector of the copula $C_{x|v}$. The copula density decomposition can be written as follows: It is obvious that $F(x_1|x_2) = h(x_1, x_2, \theta_{12})$ with $\theta_{12}$ being the
parameter (vector) of the copula \( C_{12} \). Analogously, \( F(x_3|x_2) = h(x_3, x_2, \theta_{23}) \), \( F(x_2|x_3) = h(x_2, x_3, \theta_{23}) \) and \( F(x_4|x_3) = h(x_4, x_3, \theta_{34}) \). \( F(x_1|x_2, x_3) \), again, can be iteratively simplified to
\[
\frac{\partial C_{13|2}(F(x_1|x_2), F(x_3|x_2))}{\partial F(x_3|x_2)} = h(h(x_1, x_2, \theta_{12}), h(x_3, x_2, \theta_{32}), \theta_{13|2}).
\]
Analogously, \( F(x_4|x_2, x_3) \) can be written as
\[
\frac{\partial C_{24|3}(F(x_4|x_3), F(x_2|x_3))}{\partial F(x_2|x_3)} = h(h(x_4, x_3, \theta_{43}), h(x_2, x_3, \theta_{23}), \theta_{24|3}).
\]
Finally, define \( u_1 = F(x_1), u_2 = F(x_2), u_3 = F(x_3), u_4 = F(x_4) \). The formula for the 4-dimensional PCD copula density now reads as
\[
c(u) = c_{12}(u_1, u_2) \cdot c_{23}(u_2, u_3) \cdot c_{34}(u_3, u_4)
\cdot c_{13|2}(h(u_1, u_2, \theta_{12}), h(u_3, u_2, \theta_{23})), c_{24|3}(h(u_2, u_3, \theta_{23}), h(u_4, u_3, \theta_{43}))
\cdot c_{14|23}(h(u_1, u_3, \theta_{13}), h(u_2, u_3, \theta_{23}), h(u_4, u_3, \theta_{43}), h(u_2, u_3, \theta_{23}), \theta_{24|3})).
\]
To summarize, in order to specify a \( d \)-dimensional (copula) density, two main steps have to be taken (see Aas et al., 2006): Firstly, an appropriate decomposition scheme has to be selected. Secondly, the pair-copulas have to be specified: e.g. Gaussian, Student’s \( t \), Archimedean or Gumbel copula. It is possible to use one copula model for all pair-copulas or decide individually.

### 2.3 Koehler-Symanowski (KS) copulas

Koehler & Symanowski (1995) introduce a multivariate distribution as follows: For the index set \( V = \{1, 2, \ldots, d\} \), let \( \mathcal{V} \) denote the power set of \( V \) and \( \mathcal{I} = \{I \in \mathcal{V} \text{ with } |I| \geq 2\} \).

Let further \( X \) denote a \( d \)-dimensional random vector with univariate marginal distributions \( F_i(x_i), i \in V \). For all subsets \( I \in \mathcal{I} \) let \( \alpha_I \in \mathbb{R}^+_0 \) and \( \alpha_i \in \mathbb{R}^+_0 \) for all \( i \in V \) such that \( \alpha_i = 0 + \sum_{I \in \mathcal{I}} \alpha_I > 0 \) for \( i \in I \). Then the common cdf \( F \) is defined by
\[
F(x_1, \ldots, x_d) = \prod_{I \in \mathcal{I}} \left[ \sum_{i \in I} \prod_{j \in I, j \neq i} F_j(x_j)^{\alpha_j} - (|I| - 1) \prod_{i \in I} F_i(x_i)^{\alpha_i} \right]^{|I|}.
\]

The terms \( K_I = \sum_{i \in I} \prod_{j \in I, j \neq i} F_j(x_j)^{\alpha_j} - (|I| - 1) \prod_{i \in I} F_i(x_i)^{\alpha_i} \) are called association terms. Moreover, Koehler & Symanowski (1995) showed that the joint density function exists if the marginal density functions \( f_i \) exist for all \( i \in V \). Due to the design of the Koehler-Symanowski distribution the corresponding copula has a similar functional form: Setting \( u_i = F_i(x_i) \) for all \( i \in V \), the KS copula is
\[
C(u_1, \ldots, u_d) = \prod_{I \in \mathcal{I}} \left[ \sum_{i \in I} \prod_{j \in I, j \neq i} u_j^{\alpha_j} - (|I| - 1) \prod_{i \in I} u_i^{\alpha_i} \right]^{|I|}.
\]
In contrast to the cumulative distribution function the functional representation of the density is quite complicated due to complex factors with additive components. Koehler & Symanowski (1995) gave an explicit formula for the special case of a so called KS(2)-distribution (Caputo, 1998), where all parameters $\alpha_I$ are set equal to zero for $|I| > 2$. The corresponding copula will be termed as KS(2) copula henceforth. Assuming that $\alpha_{ij} \equiv \alpha_{ji} \geq 0$ for all $(i,j) \in V \times V$ and $\alpha_{ij} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{id} > 0$ for all $i \in V$, the KS(2)-copula simplifies to

$$C(u_1, u_2, \ldots, u_d) = \prod_{i=1}^{d} u_i \prod_{i<j} K_{ij}^{-\alpha_{ij}}$$

(2.13)

with $K_{ij} = u_{i}^{1/\alpha_{i+}} + u_{j}^{1/\alpha_{j+}} - u_{i}^{1/\alpha_{i+}} u_{j}^{1/\alpha_{j+}} = K_{ji}$.

Palmitesta & Provasi (2005) apply this particular KS copula to weekly log-returns. They also argue that this copula has the ability to model complex dependence structures among subsets of marginal distribution but they do not present any goodness-of-fit measure or any comparison with other copulas.

### 2.4 Multiplicative Liebscher copulas

By now, different methods of how to construct $d$-variate copulas have been reviewed. Liebscher (2006) discusses how to combine or connect a given set of $k$ possibly different $d$-copulas $C_1, \ldots, C_k$ to a new $d$-copula $C$ in order to increase flexibility and/or introduce asymmetry. His proposal focuses on multiplicative connections of $d$-copulas of the form

$$C(u_1, \ldots, u_d) = \prod_{j=1}^{k} C_j(g_{j1}(u_1), \ldots, g_{jd}(u_d))$$

(2.14)

with a set of $k \cdot d$ admissible functions $g_{j1}, \ldots, g_{jd}, \ldots, g_{k1}, \ldots, g_{kd}$, each of which being bijective, monotonously increasing or identically equal 1 satisfying

$$\prod_{j=1}^{k} g_{ji}(v) = v, \quad i = 1, \ldots, d.$$  

(2.15)

Note that (2.15) reduces to $g_{ji}(v) = v$ for $k = 1$ and $i = 1, \ldots, d$, and $C$ is recovered. In accordance to Liebscher (2006), possible choices are

$$g_{ji}(v) = v^{\theta_{ji}} \quad \text{with } \theta_{ji} > 0 \quad \text{and} \quad \sum_{j=1}^{k} \theta_{ji} = 1 \quad \text{for } i = 1, \ldots, d$$

(2.16)

or $g_{1i}(v) = f(v)$, $g_{2i}(v) = v \cdot \frac{1}{f(v)}$, $f(v) = \left(\frac{1-e^{-\theta v}}{1-e^{-\theta}}\right)^{\alpha}$, $\theta > 0$, $\alpha \in (0,1)$.  

(2.17)
We consider four different generalized Clayton copulas based on (2.14). The "Generalized Clayton of Liebscher type I" \( (L_1) \) is obtained by setting \( k = 2 \), choosing the Clayton copula for \( C_1 \), the independence copula for \( C_2 \) and \( g_{21}(v) \) as in (2.16). Applying (2.17) rather than (2.16), the "Generalized Clayton of Liebscher type II" \( (L_2) \) copula with \( d + 2 \) dependence parameters is constructed. Similarly, combining two \( d \)-variate Clayton copulas and using \( g \) from (2.16) we obtain the \( d \)-dimensional copula family with \( d + 2 \) parameters, termed as the "Generalized Clayton of Liebscher type III" \( (L_3) \) in the sequel. Finally, applying again (2.17) rather than (2.16), the "Generalized Clayton of Liebscher type IV" \( (L_4) \) is obtained.

### 3 Goodness-of-fit measures

We now tackle the problem of comparing the goodness-of-fit (GOF) of the different copula models from section 2, noting that most of them are not nested. As we apply maximum likelihood (ML) methods to obtain estimators for the unknown parameter vector, the first choice is the log-likelihood value \( \ell \) or – in order to take the different numbers of parameters into account – the Bayesian information criterion \( BIC = -2\ell + K \ln(N) \), where \( K \) and \( N \) denote the number of parameters to be fitted and the number of observations, respectively. However, comparing log-likelihood values for non-nested models may produce misleading conclusions. Therefore, other GOF tests may come to application. Following Breymann, Dias & Embrechts (2003), Chen, Fan & Patton (2004) or recently Berg & Bakken (2006), the main idea is to project the multivariate problem onto a set of independent and uniform \( U(0,1) \) variables, given the multivariate distribution and to calculate the distance (e.g. Anderson-Darling, Kolmogorov-Smirnov, Cramér-von Mises) between the transformed variables and the uniform distribution. In contrast to the authors above, we are not primarily interested whether the data stem from the specified copula model. Instead, we use these distances as criterion itself. The proceeding is roughly as follows:

By means of the Rosenblatt (1952) transformation the random vector \( \mathbf{X} = (X_1, \ldots, X_d)' \) is mapped onto a random vector \( \mathbf{Z}^* = (Z_1^*, \ldots, Z_d^*)' \) via

\[
Z_1^* = F_1(X_1) \quad \text{and} \quad Z_i^* = F_{X_i|X_1,\ldots,X_{i-1}}(X_i|X_1,\ldots,X_{i-1}), \quad i = 2, \ldots, d. \tag{3.1}
\]

It can be shown that \( \mathbf{Z}^* \) is uniformly distributed on \([0,1]^d\) with independent components \( Z_1^*, \ldots, Z_d^* \). Assume that the cumulative distribution function of \( \mathbf{X} \) admits the decomposition

\[
F_\mathbf{X}(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)),
\]

where \( C(\cdot) \) denotes a parametric copula which is the common distribution function of \( \mathbf{U} = (U_1, \ldots, U_d)' \) with \( U_i \equiv F_{X_i}(X_i) \). Define \( C(u_1, \ldots, u_j) \equiv C(u_1, \ldots, u_j, 1, \ldots, 1) \) for \( j \leq d \). Furthermore, the conditional distribution of \( U_i|U_1, \ldots, U_{i-1} \) is given by

\[
C_i(u_i) \equiv \frac{\partial^{i-1}C(u_1, \ldots, u_i)}{\partial u_1 \ldots \partial u_{i-1}} \biggl/ \frac{\partial^{i-1}C(u_1, \ldots, u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}.
\]
According to (3.1), the variables
\[ Z_1 \equiv C(U_1) = U_1 \quad \text{and} \quad Z_i \equiv C_i(U_i), \quad i = 2, \ldots, d \] (3.2)
are independent and uniform on \([0, 1]\). Consequently, the sample \(X_1, \ldots, X_N\) from a parametric copula and with marginals given by \(F_1, \ldots, F_d\) can be mapped onto an \(iid\) sample \(Z_1, \ldots, Z_N\) from a uniform distribution on \([0, 1]^d\).

Breymann et al. (2003) suggest transforming each random vector \(Z_i = (Z_{i1}, \ldots, Z_{id})'\) into a (univariate) chi-square variable \(\chi_j\) with \(d\) degrees of freedom through \(\chi_j = \sum_{i=1}^d \Phi^{-1}(Z_{ji})^2\), \(j = 1, \ldots, N\), where \(\Phi^{-1}(u)\) denotes the standard normal quantile function. If the margins are unknown, they may be replaced by the corresponding empirical counterparts. Breymann et al. state that "we do assume that the \(\chi^2\)-distribution will not be significantly affected by the use of the empirical distribution functions used to transform the marginal data".

4 The data set

The data sets we used to compare the different copula models come from three different markets (German stock market, foreign exchange (FX) market and commodity markets). From each market, four typical representatives were selected, provided that the corresponding sample period is sufficiently large. Instead of analyzing the prices themselves, we calculated and considered (percentual) continuously compounded returns ("log-returns") \(R_t = 100(\log P_t - \log P_{t-1})\), \(t = 2, \ldots, N\). In order to account for possible time-dependencies (which are common to most financial return series), we also fitted univariate GARCH models of the form \(R_t = \mu + \gamma_1 R_{t-1} + \ldots + \gamma_k R_{t-k} + h_t \epsilon_t\) with variance equations \(h_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \ldots + \alpha_p R_{t-p}^2 + \beta_1 h_{t-1}^2 + \ldots + \beta_q h_{t-q}^2\) to each of the series and considered standardized residuals \(\epsilon_t\) rather than the original returns \(R_t\). Secondly, as we are not primarily interested in parametric models for the marginal distributions, all observations (i.e. returns or standardized residuals) were transformed into uniform ones by means of the (empirical) probability integral transform, i.e.

\[ U_t = F_N(R_t) \quad \text{with} \quad F_N(x) = \frac{\# \{R_t \leq x\}}{\# R_t} \quad \text{and} \quad U_t^{*} = F_N(\epsilon_t). \]

4.1 German stock returns

From the German stock market, we selected prices of HVB AG, BMW AG, Allianz AG and Munich Re AG, all of them being part of the German stock market index DAX which measures the performance of the Prime Standard’s 30 largest German companies in terms of order book volume and market capitalization. Figure 3 contains the series of prices and returns. Table 1 summarizes descriptive and inductive statistics. All series feature negative skewness and high kurtosis (measured by the third and fourth standardized moment \(S\) and \(K\)). Morerover, there is empirical evidence for (slight) serial correlation and GARCH effects
as the Ljung-Box statistic $LB$ and Engle's Lagrange multiplier statistic $LM$ indicate (the critical $\chi^2$-value is given by 18.307 in both cases for $\alpha = 0.05$).

<table>
<thead>
<tr>
<th>Start</th>
<th>End</th>
<th>N</th>
<th>Stocks</th>
<th>$\mu$</th>
<th>$s^2$</th>
<th>$S$</th>
<th>$K$</th>
<th>$LB(10)$</th>
<th>$LM(10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>02-01-90</td>
<td>12-11-03</td>
<td>3486</td>
<td>HVB</td>
<td>0.004</td>
<td>5.61</td>
<td>-0.033</td>
<td>8.16</td>
<td>24.45</td>
<td>621.08</td>
</tr>
<tr>
<td>02-01-90</td>
<td>12-11-03</td>
<td>3486</td>
<td>BMW</td>
<td>0.046</td>
<td>4.33</td>
<td>-0.132</td>
<td>7.19</td>
<td>28.96</td>
<td>366.49</td>
</tr>
<tr>
<td>02-01-90</td>
<td>12-11-03</td>
<td>3486</td>
<td>Allianz</td>
<td>-0.002</td>
<td>4.87</td>
<td>-0.07</td>
<td>8.37</td>
<td>29.74</td>
<td>517.14</td>
</tr>
<tr>
<td>02-01-90</td>
<td>12-11-03</td>
<td>3486</td>
<td>MunichRe</td>
<td>0.02</td>
<td>5.06</td>
<td>-0.027</td>
<td>8.75</td>
<td>50.53</td>
<td>508.59</td>
</tr>
</tbody>
</table>

Table 1: German stock returns.

4.2 Exchange rate returns

Data from foreign exchange markets (FX-markets) are available from the PACIFIC Exchange Rate Service\(^1\). This service offered by Prof. Werner Antweiler at UBC’s Sauder School of Business provides access to current and historic daily exchange rates through an on-line database retrieval and plotting system. In contrast to the volume notation, where values are expressed in units of the target currency per unit of the base currency, the price notation is used within this work which corresponds to the numerical inverse of the volume notation. All values are expressed in units of the base currency (here US-Dollar) per unit of the target currency. Table 2 summarizes the statistics of the four exchanges rates (Canadian Dollar, Japanese Yen, Swiss Franc, British Pound) which are used later on. Again, prices and log-returns are shown in figure 4, below.

\(^1\)Download under the URL-link http://pacific.commerce.ubc.ca.
Table 2: Exchange rates

<table>
<thead>
<tr>
<th>Start</th>
<th>End</th>
<th>N</th>
<th>FX Rate</th>
<th>μ</th>
<th>σ²</th>
<th>S</th>
<th>K</th>
<th>LB(10)</th>
<th>LM(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>02-01-90</td>
<td>31-12-04</td>
<td>8054</td>
<td>CAD</td>
<td>0.002</td>
<td>0.09</td>
<td>-0.004</td>
<td>6.75</td>
<td>12.65</td>
<td>912.18</td>
</tr>
<tr>
<td>02-01-90</td>
<td>31-12-04</td>
<td>8054</td>
<td>YEN</td>
<td>-0.015</td>
<td>0.56</td>
<td>-0.002</td>
<td>6.11</td>
<td>12.5</td>
<td>429.24</td>
</tr>
<tr>
<td>02-01-90</td>
<td>31-12-04</td>
<td>8054</td>
<td>SFR</td>
<td>0.003</td>
<td>0.36</td>
<td>0.132</td>
<td>6.84</td>
<td>55.79</td>
<td>485.26</td>
</tr>
<tr>
<td>02-01-90</td>
<td>31-12-04</td>
<td>8054</td>
<td>BRP</td>
<td>-0.013</td>
<td>0.44</td>
<td>-0.723</td>
<td>13.33</td>
<td>34.48</td>
<td>176.20</td>
</tr>
</tbody>
</table>

4.3 Metal returns

The London Metal Exchange\(^2\) (LME) is the world’s premier non-ferrous metals market with a turnover value of some US$2000 billion per annum. For a detailed introduction on metal markets with emphasis on the London metal exchange see Crowson & Sampson (2001). Among the different metals, emphasis is placed on aluminium, copper, lead and nickel. All prices are quoted in US-Dollar per tonne. Table 3 again contains the basic summary statistics. Prices and log-returns are displayed in figure 5.

Table 3: Metals: Prices versus Returns.

<table>
<thead>
<tr>
<th>Start</th>
<th>End</th>
<th>N</th>
<th>Metal</th>
<th>μ</th>
<th>σ²</th>
<th>S</th>
<th>K</th>
<th>LB(10)</th>
<th>LM(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>26-03-99</td>
<td>07-08-06</td>
<td>1093</td>
<td>Lead</td>
<td>0.034</td>
<td>1.22</td>
<td>-0.555</td>
<td>8.72</td>
<td>29.74</td>
<td>161.56</td>
</tr>
<tr>
<td>26-03-99</td>
<td>07-08-06</td>
<td>1093</td>
<td>Tin</td>
<td>0.084</td>
<td>4.21</td>
<td>-0.368</td>
<td>5.59</td>
<td>30.11</td>
<td>100.37</td>
</tr>
<tr>
<td>26-03-99</td>
<td>07-08-06</td>
<td>1093</td>
<td>Nickel</td>
<td>0.142</td>
<td>2.38</td>
<td>-0.139</td>
<td>5.13</td>
<td>12.95</td>
<td>127.66</td>
</tr>
<tr>
<td>26-03-99</td>
<td>07-08-06</td>
<td>1093</td>
<td>Zinc</td>
<td>0.13</td>
<td>4.97</td>
<td>-0.618</td>
<td>7.9</td>
<td>10.72</td>
<td>21.45</td>
</tr>
</tbody>
</table>

\(^2\)Download under http://www.lme.co.uk/.

14
5 Empirical results

The 4-copulas under consideration are the following: Firstly, we selected the Clayton copula (CLA), the Gumbel copula (GUM) and its rotated version (roGUM) from the Archimedean class. From the generalized Archimedean copula family, two hierarchical copula models (i.e. HA-CLA and HA-GUM) are included, based on the Clayton and the Gumbel copula, respectively. Moreover, six representatives of Morrillas' construction scheme (i.e. MO-CLA1, MO-CLA2, MO-CLA3, MO-GUM1, MO-GUM2, MO-GUM3) involving the Clayton, the Gumbel and different generator functions (no. 3, 2, 4 in Morillas, 2005) are included as well. In addition, two version of Liebscher’s proposal (GMLF, GML2) are used. Beyond that, representing the "elliptical copula world", the Gaussian copula (NORM) and – as ultimate benchmark – the Student-t (T) copula are also included. From the pair-copula decomposition we chose five representatives (i.e. PC-NORM, PC-T, PC-CLA, PC-GUM, PC-roGUM) each of them derived from one single copula model (i.e. we used no decompositions based on different copulas). Additionally, we fit the KS(2)-copula of Palmitesta & Provasi (2005) as well as two generalized versions (i.e the augmented KS(2)-copula, denoted by aKS(2), where the four-dimensional association parameter is added and the fully specified model, briefly KSC) of Koehler & Symanowski (1995). Finally, four different types of multiplicative Liebscher copulas from example 2.8 ($L_1$, $L_2$, $L_3$, $L_4$) are considered.

The computer code for the ML-estimation was implemented in Matlab 7.1. For maximization purposes we used the line-search algorithm of Matlab. We calculated parameter estimates and their standard errors as well as the different goodness-of-fit measures for the

---

*It should be pointed out that the standard errors in the subsequent tables (which we extracted from the estimated Fisher information) should be interpreted as "rough-and-ready" estimators for the true (unknown) ones. As we make use of the semi-parametric estimation procedure (where the unknown marginals are replaced by the empirical distribution function), the correct choice would be to adopt the proposal of Genest, Ghoudi & Rivest (1995). Because we are primarily interested in comparing different goodness-of-fit
For Peer Review Only

GARCH-residuals and all copula models mentioned above. As stated above, goodness-of-fit is measured by the Log-likelihood value and the BIC criterion. Above that, three distance measures,

\[
\text{KS} = \sqrt{N} \max_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right|,
\]

\[
\text{AKS} = \frac{1}{\sqrt{N}} \sum_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right| \quad \text{and} \quad \mathcal{L}_2 = \left\| F_{\chi^2(d)} - F_{N,\chi} \right\|_2
\]

are calculated to quantify the distance after application of the Rosenblatt transformation (based on the different parametric copula models).

The subsequent tables summarize the estimation results for the different copula models under consideration. In contrast to table 5,6 and 7 which are dedicated to the parameter estimates and their approximate standard error, table 4 displays five goodness-of-fit measures for every copula and every data set. As already mentioned above, we only presented the results for the GARCH residuals, emphasizing that using the original data instead doesn’t change the estimation results substantially. In particular, the ordering of the goodness-of-fit measures is essentially preserved. Above that, parameter estimates of the dependence parameters are roughly stable for most of the copulas under consideration. In general, the results in 4 seem to be rather stable across all data sets and distance measures. The most important conclusions are the following:

First of all, both Student-t copula (T) and the pair copula built from bivariate Student-t copulas (PC-T) provide the best fit over all measures. Taking the comparatively large number of parameters of the PC-T into account, the Student-t copula should be preferred from a practical point of view.

Secondly, within the class of pair-copulas itself, the pair-copula approach based on different bivariate rotated Gumbel copulas dominates the approaches based on bivariate Clayton and bivariate Gumbel copulas, though being outperformed by the above-mentioned PC-t approach. Among the different construction schemes of multivariate copulas, the pair-copula approach has to be pointed out.

Thirdly, the Gaussian copula seems to attain more attraction if the number of dimensions increases. Whereas Archimedean copulas frequently outperform this dependence model in the bivariate case, the situation seems to reverse for the higher-dimensional case. There is empirical evidence that the overall goodness-of-fit of the Gaussian copula (measured by the log-likelihood) seems to be very good, whereas distance measures (with more emphasis on the tail area) noticeably worsen compared to other copulas.

Fourthly, the KS(2)-copula which was advocated by Palmitesta and Provasi (2005) provides only a poor fit to the return series. In contrast to the original specification of Koehler and Symanowski (1995), it neglects three and four-dimensional association parameters. Taking measures and not in checking significances, this procedure (which requires derivatives of the log-density of the underlying copula models) is omitted.
For Peer Review Only

a global (i.e. four-dimensional) association parameter into account, the corresponding augmented KS copula (aKS(2)) clearly improves all goodness-of-fit measures (Note that most of distance measures cut in half) which can be further improved if the fully specified KS copula (KSC) is used. The latter proves to be a serious alternative to the elliptical competitors, at least in the 4-dimensional case.

Fifth, the fit of "plain" Archimedean copulas significantly improves if a generalized model based on the proposals of Liebscher and Morillas is taken into consideration. Within both classes, there seems to be further discussion as to how to choose the underlying generating functions. Basically, the multiplicative Liebscher copulas $L_1 - L_4$ as well as GML2 and GLMF tend to outperform the representatives of Morillas’ class.

Sixth, focusing on hierarchical Archimedean copula families, we found only slight improvement regarding the goodness-of-fit, at least for our data sets. However, we admit that one might further improve the results with another hierarchy which might be found on the basis of cluster algorithms.

To sum up, the 4-variate Student-$t$ distribution still plays a predominant role. Some of the recently proposed construction schemes are partially competitive while others are more likely to be overestimated in the relevant literature. Finally, our findings are derived from 4-dimensional data sets. For the higher dimensional case, some results are expected to become still more evident, while some of the models under consideration (e.g. KSC) will be no longer estimable.

Acknowledgement

The authors thank two anonymous referees for their helpful comments and suggestions which significantly improved the presentation of the paper.
<table>
<thead>
<tr>
<th>Copula</th>
<th>German stocks (GARCH-Residuals)</th>
<th>Exchange rates (GARCH-Residuals)</th>
<th>Metal data (GARCH-Residuals)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>l</td>
<td>BIC</td>
<td>KS</td>
</tr>
<tr>
<td></td>
<td>1777.9</td>
<td>-3550.5</td>
<td>5.49</td>
</tr>
<tr>
<td></td>
<td>1786.0</td>
<td>-3563.1</td>
<td>4.64</td>
</tr>
<tr>
<td></td>
<td>1818.6</td>
<td>-3628.1</td>
<td>4.73</td>
</tr>
<tr>
<td></td>
<td>3564.5</td>
<td>-7075.1</td>
<td>4.44</td>
</tr>
<tr>
<td></td>
<td>3872.6</td>
<td>-7822.2</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td>3564.4</td>
<td>-7074.8</td>
<td>4.45</td>
</tr>
<tr>
<td></td>
<td>3958.3</td>
<td>-7808.6</td>
<td>1.74</td>
</tr>
<tr>
<td></td>
<td>2950.9</td>
<td>-5847.9</td>
<td>5.20</td>
</tr>
<tr>
<td></td>
<td>3407.1</td>
<td>-6760.2</td>
<td>3.56</td>
</tr>
<tr>
<td></td>
<td>3561.5</td>
<td>-7068.9</td>
<td>3.14</td>
</tr>
<tr>
<td></td>
<td>2711.8</td>
<td>-5369.6</td>
<td>5.64</td>
</tr>
<tr>
<td></td>
<td>2712.2</td>
<td>-5379.5</td>
<td>5.52</td>
</tr>
<tr>
<td></td>
<td>164.7</td>
<td>-239.4</td>
<td>6.84</td>
</tr>
<tr>
<td></td>
<td>3440.9</td>
<td>-6783.0</td>
<td>2.76</td>
</tr>
<tr>
<td></td>
<td>3540.0</td>
<td>-6945.1</td>
<td>2.15</td>
</tr>
<tr>
<td></td>
<td>1967.3</td>
<td>-3916.6</td>
<td>4.97</td>
</tr>
<tr>
<td></td>
<td>1991.6</td>
<td>-3965.2</td>
<td>4.37</td>
</tr>
<tr>
<td></td>
<td>1779.7</td>
<td>-3541.5</td>
<td>5.49</td>
</tr>
<tr>
<td></td>
<td>1786.0</td>
<td>-3554.1</td>
<td>4.64</td>
</tr>
<tr>
<td></td>
<td>1967.3</td>
<td>-3916.6</td>
<td>4.97</td>
</tr>
<tr>
<td></td>
<td>1991.6</td>
<td>-3965.2</td>
<td>4.37</td>
</tr>
<tr>
<td></td>
<td>3044.9</td>
<td>-6044.8</td>
<td>4.58</td>
</tr>
<tr>
<td></td>
<td>3104.9</td>
<td>-6155.8</td>
<td>3.42</td>
</tr>
<tr>
<td></td>
<td>2399.0</td>
<td>-4744.0</td>
<td>5.14</td>
</tr>
<tr>
<td></td>
<td>2278.2</td>
<td>-4493.5</td>
<td>2.99</td>
</tr>
<tr>
<td></td>
<td>1805.3</td>
<td>-3583.5</td>
<td>5.36</td>
</tr>
<tr>
<td></td>
<td>1913.6</td>
<td>-3800.3</td>
<td>4.38</td>
</tr>
</tbody>
</table>

Table 4: Goodness-of-fit measures: German stock returns (left), Exchange rates (middle) and Metal returns (right)
<table>
<thead>
<tr>
<th>Copula</th>
<th>θ</th>
<th>Copula</th>
<th>θ</th>
<th>θ₃</th>
<th>θ₄</th>
<th>θ₅</th>
<th>θ₆</th>
<th>Copula</th>
<th>θ₁₁</th>
<th>θ₁₂</th>
<th>θ₂₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLA</td>
<td>0.6145 (0.014)</td>
<td>PC-CLA</td>
<td>0.5509 (0.0275)</td>
<td>0.6899 (0.0294)</td>
<td>0.9086 (0.0324)</td>
<td>0.5524 (0.0312)</td>
<td>0.1453 (0.0201)</td>
<td>0.0724 (0.0181)</td>
<td>HA-CLA</td>
<td>0.5084 (0.0261)</td>
<td>0.8856 (0.0314)</td>
</tr>
<tr>
<td>GUM</td>
<td>1.3820 (0.0099)</td>
<td>PC-GUM</td>
<td>1.3158 (0.0162)</td>
<td>1.4146 (0.0168)</td>
<td>1.5678 (0.0201)</td>
<td>1.3537 (0.0185)</td>
<td>1.0658 (0.0104)</td>
<td>1.0468 (0.0099)</td>
<td>HA-GUM</td>
<td>1.3818 (0.0175)</td>
<td>1.5794 (0.0197)</td>
</tr>
<tr>
<td>roGUM</td>
<td>1.3866 (0.0099)</td>
<td>PC-roGUM</td>
<td>1.3437 (0.0174)</td>
<td>1.4444 (0.0192)</td>
<td>1.5898 (0.0211)</td>
<td>1.3741 (0.0194)</td>
<td>1.0742 (0.0115)</td>
<td>1.0334 (0.0103)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ρ₁₂</td>
<td>ρ₁₃</td>
<td>ρ₁₄</td>
<td>ρ₂₃</td>
<td>ρ₂₄</td>
<td>ρ₂₅</td>
<td>ρ₂₆</td>
<td></td>
<td>ρ₃₁</td>
<td>ρ₃₂</td>
<td>ρ₃₃</td>
</tr>
<tr>
<td>NORM</td>
<td>0.4220 (0.0129)</td>
<td>0.5558 (0.0102)</td>
<td>0.3985 (0.0129)</td>
<td>0.4888 (0.0116)</td>
<td>0.3729 (0.0113)</td>
<td>0.5770</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4327 (0.0129)</td>
<td>0.5715 (0.0102)</td>
<td>0.4097 (0.0129)</td>
<td>0.5046 (0.0115)</td>
<td>0.3831 (0.0113)</td>
<td>0.5785 (0.0101)</td>
<td>10.0271 (0.8325)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4219 (0.0129)</td>
<td>0.4887 (0.0117)</td>
<td>0.5768 (0.0104)</td>
<td>0.4420 (0.0113)</td>
<td>0.1276 (0.017)</td>
<td>0.0006 (0.0174)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4330 (0.0043)</td>
<td>0.5047 (0.0027)</td>
<td>0.5816 (0.0046)</td>
<td>0.4565 (0.0027)</td>
<td>0.1307 (0.0043)</td>
<td>0.0945 (0.0018)</td>
<td>8.3406 (1.2041)</td>
<td>6.9444 (1.4645)</td>
<td>8.8916 (1.7178)</td>
<td>8.2465 (1.2688)</td>
<td>42.3716 (12.0791)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>σ₁₁</td>
<td>σ₁₂</td>
<td>σ₁₃</td>
<td>σ₁₄</td>
<td>σ₂₂</td>
<td>σ₂₃</td>
<td>σ₂₄</td>
<td>σ₂₅</td>
<td>σ₂₆</td>
<td>σ₃₁</td>
<td>σ₃₂</td>
</tr>
<tr>
<td>KS(2)</td>
<td>0.0231 (0.0076)</td>
<td>0.077 (0.0213)</td>
<td>0.2173 (0.0186)</td>
<td>0.0631 (0.0177)</td>
<td>0.1912 (0.0234)</td>
<td>0.0000</td>
<td>0.0000 (0.0072)</td>
<td>0.171 (0.098)</td>
<td>0.0000 (0.0057)</td>
<td>0.6181 (0.0316)</td>
<td></td>
</tr>
<tr>
<td>aKS(2)</td>
<td>0.0751 (0.0126)</td>
<td>0.0465 (0.0013)</td>
<td>0.1016 (0.0095)</td>
<td>0.0206 (0.0085)</td>
<td>0.1465 (0.0181)</td>
<td>0.0545</td>
<td>0.0159 (0.0102)</td>
<td>0.002 (0.002)</td>
<td>0.113 (0.0160)</td>
<td>0.1098 (0.0184)</td>
<td>0.2347 (0.0173)</td>
</tr>
<tr>
<td>KSC</td>
<td>0.0586 (0.0142)</td>
<td>0.0317 (0.0113)</td>
<td>0.0393 (0.0038)</td>
<td>0.0046 (0.0097)</td>
<td>0.1021 (0.0162)</td>
<td>0.0074</td>
<td>0.0059 (0.0011)</td>
<td>0.0051 (0.0018)</td>
<td>0.0526 (0.0058)</td>
<td>0.0892 (0.0113)</td>
<td>0.0874 (0.0071)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>γ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ₁</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ₂</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ₃</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ₄</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ₅</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: German stock returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)
Table 6: Exchange rate returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)
<table>
<thead>
<tr>
<th>Copula</th>
<th>θ_1</th>
<th>θ_2</th>
<th>θ_3</th>
<th>θ_4</th>
<th>θ_5</th>
<th>θ_6</th>
<th>Copula</th>
<th>θ_11</th>
<th>θ_12</th>
<th>θ_21</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLA</td>
<td>0.5773</td>
<td>0.3967</td>
<td>0.8440</td>
<td>0.7863</td>
<td>0.4752</td>
<td>0.2084</td>
<td>HA-CLA</td>
<td>0.5528</td>
<td>0.7627</td>
<td>0.5527</td>
</tr>
<tr>
<td>GUM</td>
<td>1.3286</td>
<td>1.2022</td>
<td>1.4919</td>
<td>1.2639</td>
<td>1.1250</td>
<td>1.0226</td>
<td>HA-GUM</td>
<td>1.3126</td>
<td>1.4356</td>
<td>1.3156</td>
</tr>
<tr>
<td>roGUM</td>
<td>1.3369</td>
<td>1.2323</td>
<td>1.5928</td>
<td>1.4799</td>
<td>1.2952</td>
<td>1.1339</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Copula</th>
<th>ρ_12</th>
<th>ρ_13</th>
<th>ρ_14</th>
<th>ρ_23</th>
<th>ρ_24</th>
<th>ν_1</th>
<th>ν_2</th>
<th>ν_3</th>
<th>ν_4</th>
<th>ν_5</th>
<th>ν_6</th>
</tr>
</thead>
<tbody>
<tr>
<td>NORM</td>
<td>0.3222</td>
<td>0.4783</td>
<td>0.3094</td>
<td>0.5536</td>
<td>0.4333</td>
<td>0.5104</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>0.3271</td>
<td>0.4818</td>
<td>0.3159</td>
<td>0.5613</td>
<td>0.4367</td>
<td>0.5174</td>
<td>18.8384</td>
<td>(4.4583)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PC-NORM</td>
<td>0.3215</td>
<td>0.5528</td>
<td>0.5098</td>
<td>0.3814</td>
<td>0.2106</td>
<td>0.0732</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PC-T</td>
<td>0.3301</td>
<td>0.5634</td>
<td>0.5209</td>
<td>0.3819</td>
<td>0.2110</td>
<td>0.0744</td>
<td>27.4447</td>
<td>(19.344)</td>
<td>11.6176</td>
<td>13.9478</td>
<td>12.2527</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Copula</th>
<th>θ</th>
<th>γ</th>
<th>Copula</th>
<th>θ_1</th>
<th>θ_2</th>
<th>θ_3</th>
<th>θ_4</th>
<th>θ_5</th>
<th>θ_6</th>
</tr>
</thead>
<tbody>
<tr>
<td>KS(2)</td>
<td>0.3199</td>
<td>0.0000</td>
<td>0.0174</td>
<td>0.0010</td>
<td>0.0049</td>
<td>0.0108</td>
<td>0.0604</td>
<td>0.0000</td>
<td>0.0897</td>
</tr>
<tr>
<td>aKS(2)</td>
<td>0.2190</td>
<td>0.0053</td>
<td>0.0606</td>
<td>0.0011</td>
<td>0.0071</td>
<td>0.1251</td>
<td>0.0570</td>
<td>0.0000</td>
<td>0.0817</td>
</tr>
<tr>
<td>KSC</td>
<td>0.1709</td>
<td>0.9034</td>
<td>0.0289</td>
<td>0.0047</td>
<td>0.0716</td>
<td>0.0011</td>
<td>0.0040</td>
<td>0.0000</td>
<td>0.0273</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Copula</th>
<th>θ</th>
<th>r</th>
<th>Copula</th>
<th>γ</th>
<th>θ_1</th>
<th>θ_2</th>
<th>θ_3</th>
<th>θ_4</th>
<th>θ_5</th>
<th>θ_6</th>
</tr>
</thead>
<tbody>
<tr>
<td>MO-CLA1</td>
<td>0.2649</td>
<td>-2.7984</td>
<td>L1</td>
<td>1.2887</td>
<td>0.5963</td>
<td>0.8355</td>
<td>0.9115</td>
<td>0.6852</td>
<td>0.0478</td>
<td></td>
</tr>
<tr>
<td>MO-GUM1</td>
<td>1.0554</td>
<td>-4.1123</td>
<td>L2</td>
<td>3.3950</td>
<td>0.7079</td>
<td>0.1801</td>
<td>0.598</td>
<td>0.4441</td>
<td>0.3642</td>
<td></td>
</tr>
<tr>
<td>MO-CLA2</td>
<td>10.6418</td>
<td>18.4340</td>
<td>L3</td>
<td>1.6396</td>
<td>0.3698</td>
<td>0.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7786</td>
<td></td>
</tr>
<tr>
<td>MO-GUM2</td>
<td>1.3286</td>
<td>2.0008</td>
<td>L4</td>
<td>3.1137</td>
<td>0.4960</td>
<td>0.4821</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>MO-CLA3</td>
<td>0.2649</td>
<td>0.7367</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MO-GUM3</td>
<td>1.0554</td>
<td>0.8044</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Copula</th>
<th>δ_1</th>
<th>δ_2</th>
<th>δ_3</th>
<th>δ_4</th>
<th>θ_1</th>
<th>θ_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMLF</td>
<td>1.2633</td>
<td>1.9066</td>
<td>2.0000</td>
<td>1.6572</td>
<td>2.0387</td>
<td>0.3132</td>
</tr>
<tr>
<td>GML2</td>
<td>1.2470</td>
<td>1.9057</td>
<td>2.0000</td>
<td>1.6499</td>
<td>1.1666</td>
<td>0.0978</td>
</tr>
</tbody>
</table>

Table 7: Metal returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)
References


