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Update rules for convex risk measures

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Abstract

In the first part of the paper we investigate properties that describe the intertemporal structure of dynamic convex risk measures. The usual backward approach to dynamic risk assessment leads to strong and weak versions of time consistency. As an alternative, we introduce a forward approach of consecutivity.

In the second part we discuss the problem of how to update a convex risk measure when new information arrives. We analyse to what extent the above properties are appropriate update criteria.

1 Introduction

The need of banks and insurance companies to quantify the risk of their financial positions in monetary units has motivated a systematic analysis of risk measurement in Mathematical Finance. The axiomatic approach to coherent risk measures and their acceptance sets was initiated by Artzner, Delbaen, Eber & Heath [1]. Föllmer & Schied [9], [10] and Frittelli & Rosazza Gianin [12] extended that approach by investigating the structure of convex risk measures.

A financial position is described by a random variable X on a measurable space $(\Omega, \mathcal{F})$ usually equipped with some probability measure. A convex risk measure ρ is defined as a monotone, cash invariant and convex functional on a whole class of positions. The real value $\rho(X)$ is interpreted as the minimal amount of money that has to be added to X and invested in a risk-free manner in order to make the position acceptable.

In recent years, conditional and dynamic risk measures have been studied which also take into account the role of information. We refer to, e. g., Artzner, Delbaen, Eber, Heath & Ku [2], Riedel [15] and Roorda & Schumacher [16] for the coherent case, and to Detlefsen & Scandolo [5], Klöppel & Schweizer [14], Weber [21], Föllmer & Penner [8] and Cheridito, Delbaen & Kupper [3] for the general convex case.

Suppose that at the time of evaluation a certain information is available, and let this be represented by a σ-field $\mathcal{G} \subset \mathcal{F}$. A conditional risk measure $\rho_{\mathcal{G}}$ assigns to every financial position X a $\mathcal{G}$-measurable random variable $\rho_{\mathcal{G}}(X)$. As in the unconditional case above, this random variable can be regarded as a capital requirement. A dynamic risk measure is a family of conditional risk measures adapted to a filtration that describes the information

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structure in the course of time. Conditional risk measures are by now well understood. In today's literature, the focus is on dynamic risk measures and how risk evaluations at different times should be related.

In this paper we discuss the question of dynamic risk assessment in a simple setting. We consider a dynamic risk measure $(\rho, \rho_{\mathcal{G}})$ consisting of an unconditional and a conditional risk measure, i. e., information is represented by a filtration $\{\emptyset, \Omega\} \subset \mathcal{G} \subset \mathcal{F}$.

In the usual backward approach, diffent notions of time consistency arise, which capture the idea that certain risk relations reflected by $\rho_{\mathcal{G}}$ should also be imposed on ρ. Strong time consistency, which has been studied by, e. g., [2] and [15] in the coherent case, and [5], [14], [8] and [3] in the convex case, is equivalent to the recursive equation $\rho(X) = \rho(-\rho_{\mathcal{G}}(X))$. Weakening of the strict equality to an inequality $\rho(X + \rho_{\mathcal{G}}(X)) \le / \ge 0$ leads to weak acceptance and rejection consistency. These properties have been analysed by [21] and [16] for distribution invariant and coherent risk measures. In section 3 we compare and generalise the above notions of time consistency by introducing a test set, which we use to refine the degree of consistency.

Dynamic risk assessment can also be considered from another perspective. Suppose that the information structure is not known completely in advance. Then the convex risk measure ρ has to be modified or updated under appropriate criteria when new information $\mathcal{G} \subset \mathcal{F}$ arrives. In our opinion, updating demands an approach that is rather forward-directed in time. However, in the literature on dynamic risk measures this aspect has not yet been taken into account. In section 4 we therefore suggest a notion of consecutivity. There the conditional risk measure $\rho_{\mathcal{G}}$ is regarded as a consequence of ρ in the sense that it is compatible with the initial risk assessment and the incoming information.

In the linear case, where the risk measure ρ is given by the expectation w. r. t. some probability measure and the conditional risk measure $\rho_{\mathcal{G}}$ by the conditional expectation, backward and forward approach are equivalent. In the general convex case they are different. This is illustrated in section 5 by the class of robust shortfall risk measures.

The second part of the paper deals with the updating problem. We study update rules for convex and coherent risk measures and analyse three different update criteria. We show in section 6 that strong time consistency is not appropriate as an update criterion because it is far too restrictive. We prove that a convex risk measure, which admits a strongly time consistent update rule, is uniquely determined by its values on simple positions of the form $x\mathbb{I}_A$. In the coherent case, existence of such an update rule even implies linearity of the risk measure. Similar results concerning strong time consistency can be found in the economic literature. There the updating problem has been discussed in the context of non-additive set functions and preferences, cf. Dempster [4] and Shafer [18], Walley [20], Fagin & Halpern [7], Yoo [22], Gilboa & Schmeidler [13] and Epstein & Le Breton [6].

In contrast to this, weak acceptance consistency can be used to define update rules. We show this in section 7. However, under this criterion, there may exist financial positions, which are accepted by the initial risk measure ρ no matter which $\mathcal{G}$-measurable event actually occurs in the future, and which are then rejected by the conditional risk measure $\rho_{\mathcal{G}}$. This means that new information may cause a decrease of risk tolerance, which does not seem reasonable from an economic point of view. In order to overcome this drawback, we suggest in section 8 to apply the forward-directed criterion of consecutivity.

The paper is organised as follows. In section 2 we formulate the updating problem and provide basic definitions. Then we discuss intertemporal relations between an unconditional and a conditional risk measure. Section 3 deals with time consistency. In section 4 we introduce the notion of consecutivity. In section 5 we illustrate these properties by considering a robust shortfall risk measure. In section 6, 7 and 8 we analyse to what extent a convex risk measure can be updated under the respective criterion of strong time consistency, weak acceptance consistency and consecutivity.

2 Convex risk measures and their updating

We consider a probability space $(\Omega, \mathcal{F}, P)$ where P is a reference measure. All inequalities and equalities applied to random variables are understood to hold P-almost surely. The set of financial positions is given by $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, P)$. The information which is available when the risk of those financial positions is evaluated is represented by a σ-field $\mathcal{G} \subset \mathcal{F}$. We set $\mathcal{X}_\mathcal{G} := L^\infty(\Omega, \mathcal{G}, P)$ and define conditional convex risk measures as in [5]:

Definition 2.1 *A mapping $\rho_\mathcal{G} : \mathcal{X} \to \mathcal{X}_\mathcal{G}$ is called a* conditional convex risk measure *w.r.t. $\mathcal{G} \subset \mathcal{F}$ if it satisfies the following properties for all $X, Y \in \mathcal{X}$:*

(i) Monotonicity: $\rho_\mathcal{G}(X) \geq \rho_\mathcal{G}(Y)$ whenever $X \leq Y$.

(ii) Translation invariance: $\rho_\mathcal{G}(X + X') = \rho_\mathcal{G}(X) - X'$ for all $X' \in \mathcal{X}_\mathcal{G}$.

(iii) Normalisation: $\rho_\mathcal{G}(0) = 0$.

(iv) Conditional convexity: $\rho_\mathcal{G}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_\mathcal{G}(X) + (1-\lambda)\rho_\mathcal{G}(Y)$ for all $\lambda \in \mathcal{X}_\mathcal{G}$, $0 \leq \lambda \leq 1$.

A conditional convex risk measure $\rho_\mathcal{G}$ is called a conditional coherent risk measure *if it satisfies:*

(v) Conditional positive homogeneity: $\rho_\mathcal{G}(\lambda X) = \lambda \rho_\mathcal{G}(X)$ for all $\lambda \in \mathcal{X}_\mathcal{G}$, $\lambda \geq 0$.

It is well known that there is a 1-1-correspondence between conditional convex risk measures $\rho_\mathcal{G}$ and their conditionally convex acceptance sets $\mathcal{A}_\mathcal{G} := \{X \in \mathcal{X} \mid \rho_\mathcal{G}(X) \leq 0\}$. Every risk measure $\rho_\mathcal{G}$ can be recovered from its acceptance set $\mathcal{A}_\mathcal{G}$ via

$$\rho_\mathcal{G}(X) = \operatorname{ess\,inf}\{X' \in \mathcal{X}_\mathcal{G} \mid X + X' \in \mathcal{A}_\mathcal{G}\} \quad \forall\, X \in \mathcal{X}. \tag{1}$$

Thus, the random variable $\rho_\mathcal{G}(X)$ can be regarded as a conditional capital requirement that is needed to make the position X acceptable. For a proof of the above representation and for further properties of the acceptance set $\mathcal{A}_\mathcal{G}$, we refer the interested reader to [5], proposition 2.5, and to [14].

It is also possible to define a rejection set $\mathcal{N}_\mathcal{G} := \{X \in \mathcal{X} \mid \rho_\mathcal{G}(X) \geq 0\}$. Then we have

$$\rho_\mathcal{G}(X) = \operatorname{ess\,sup}\{X' \in \mathcal{X}_\mathcal{G} \mid X + X' \in \mathcal{N}_\mathcal{G}\} \quad \forall\, X \in \mathcal{X}.$$

Note that every financial position $X \in \mathcal{X}$ with $\rho_\mathcal{G}(X) = 0$ belongs to the rejection set which slightly conflicts with the interpretation of $\rho_\mathcal{G}(X)$ as a capital requirement. Moreover, in general, conditional convexity of $\rho_\mathcal{G}$ is not reflected by the rejection set.

The problem that we are going to discuss is the following. Suppose, initially, an unconditional convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is used for risk evaluation, i. e., the available information is trivial. Then new information arrives. It is represented by a σ-field $\mathcal{G} \subset \mathcal{F}$. The question arises how the convex risk measure ρ should be modified according to this information. To answer this question, we study reasonable update criteria.

Given some σ-field $\mathcal{G} \subset \mathcal{F}$, let us denote by $\mathcal{R}_\mathcal{G}$ the class of all conditional convex risk measures $\rho_\mathcal{G} : \mathcal{X} \to \mathcal{X}_\mathcal{G}$.

Definition 2.2 *An* update rule *for a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is a mapping that assigns to every σ-field $\mathcal{G} \subset \mathcal{F}$ a conditional convex risk measure $\rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}$ such that $\rho_{\{\emptyset,\Omega\}} = \rho$.*

Under full information there exists only one conditional risk measure $\rho_\mathcal{F} = -\mathrm{id}_\mathcal{X}$. Therefore, we assume throughout this paper that $(\Omega, \mathcal{F}, P)$ contains at least three disjoint events which have positive probability under the reference measure P. Otherwise the updating problem would be trivial.

We analyse to what extent the backward conditions of time consistency, which are usually applied to define risk measures in a dynamic setting, are appropriate update criteria. We also consider a forward condition of consecutivity where each conditional risk measure is compatible with the initial risk assessment and the incoming information.

For a fixed σ-field $\mathcal{G} \subset \mathcal{F}$, an update criterion may be satisfied by a whole class of conditional risk measures. Then we use a partial order to compare the alternatives.

Definition 2.3 *Let $\rho_\mathcal{G}, \rho'_\mathcal{G} \in \mathcal{R}_\mathcal{G}$ be two conditional convex risk measures. We write*

$$\rho_\mathcal{G} \preceq \rho'_\mathcal{G} \quad :\Leftrightarrow \quad \mathcal{A}_\mathcal{G} \subset \mathcal{A}'_\mathcal{G}, \tag{2}$$

and we say that $\rho_\mathcal{G}$ is less tolerant *than $\rho'_\mathcal{G}$.*

Recall that $\rho_\mathcal{G} \in \mathcal{R}'_\mathcal{G}$ is the least element of a class $\mathcal{R}'_\mathcal{G} \subset \mathcal{R}_\mathcal{G}$ w. r. t. the partial order if $\rho_\mathcal{G} \preceq \rho'_\mathcal{G}$ for all $\rho'_\mathcal{G} \in \mathcal{R}'_\mathcal{G}$. Then we say that $\rho_\mathcal{G}$ is least tolerant in $\mathcal{R}'_\mathcal{G}$. On the other hand, $\rho_\mathcal{G}$ is a minimal element if $\rho'_\mathcal{G} \preceq \rho_\mathcal{G}$ implies $\rho'_\mathcal{G} \simeq \rho_\mathcal{G}$ for all $\rho'_\mathcal{G} \in \mathcal{R}'_\mathcal{G}$. That means, there is no other risk measure in $\mathcal{R}'_\mathcal{G}$ that is less tolerant than $\rho_\mathcal{G}$. The definition of the greatest element and maximality is analogous.

Remark 2.4 *The partial order defined in (2) satisfies the following conditions:*

(i) $\rho_\mathcal{G} \preceq \rho'_0$ iff $\rho_\mathcal{G}(X) \geq \rho'_\mathcal{G}(X)$ for all $X \in \mathcal{X}$.

(ii) The worst-case risk measure $\rho^{\mathrm{worst}}_\mathcal{G}$ with $\mathcal{A}^{\mathrm{worst}}_\mathcal{G} = \mathcal{X}_+$ is least tolerant in $\mathcal{R}_\mathcal{G}$.

(iii) A conditional linear risk measure

$$\rho_\mathcal{G}(X) := \mathbb{E}_Q(-X|\mathcal{G}) \quad \forall\, X \in \mathcal{X},$$

generated by an equivalent probabilty measure $Q \sim P$ satisfies maximality. Indeed, if $\rho_\mathcal{G} \preceq \rho'_\mathcal{G} \in \mathcal{R}_\mathcal{G}$, then normalisation and convexity of $\rho'_\mathcal{G}$ (applied to $X, -X$ and $\lambda = 1/2$) yield

$$-\mathbb{E}_Q(X|\mathcal{G}) \leq -\rho'_\mathcal{G}(-X) \leq \rho'_\mathcal{G}(X) \leq \mathbb{E}_Q(-X|\mathcal{G}) \quad \forall\, X \in \mathcal{X}.$$

Hence, $\rho_\mathcal{G} \simeq \rho'_\mathcal{G}$, both risk measures are equally tolerant.

In order to characterise and to compare update rules for a given convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$, we assume throughout this paper that ρ admits a robust representation in terms of equivalent probability measures:

$$\rho(X) = \sup \{\mathbb{E}_Q(-X) - \alpha_{\min}(Q) \mid Q \sim P\} \quad \forall\, X \in \mathcal{X}, \tag{3}$$

where $\alpha_{\min}$ is the so-called minimal penalty function defined by

$$\alpha_{\min}(Q) := \sup_{X \in \mathcal{A}} \mathbb{E}_Q(-X) \quad \forall\, Q \sim P.$$

Under the necessary condition of continuity from above such a robust representation exists if ρ is sensitive, i. e.,

$$P(A) > 0 \quad \Rightarrow \quad \rho(-\varepsilon \mathbb{I}_A) > 0 \quad \forall\, \varepsilon > 0 \tag{4}$$

for every event $A \in \mathcal{F}$, cf. [8]. By normalisation of ρ, we have

$$\inf_{Q \sim P} \alpha_{\min}(Q) = 0.$$

We assume moreover that there exists at least one equivalent probabilistic model which is not penalised under the minimal penalty function of ρ, i. e.,

$$\exists\, Q \sim P \quad \text{with} \quad \alpha_{\min}(Q) = 0. \tag{5}$$

Note that a convex risk measure which admits a robust representation as in (3) and (5) is sensitive as in (4).

3 Time consistency

Let us fix a σ-field $\mathcal{G} \subset \mathcal{F}$. We consider a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ and a conditional convex risk measure $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$. In this section we study different versions of time consistency. These are intertemporal relations between ρ and $\rho_{\mathcal{G}}$ which correspond to an approach to dynamic risk assessment that is backward-directed in time.

We choose a set $\mathcal{Y} \subset \mathcal{X}$ of financial positions such that $0 \in \mathcal{Y}$ and $\mathcal{Y} + \mathbb{R} = \mathcal{Y}$. This test set determines the degree of time consistency. There are three important examples that we want to study in detail: $\mathcal{Y} = \mathbb{R}$, $\mathcal{Y} = \mathcal{X}_{\mathcal{G}}$ and $\mathcal{Y} = \mathcal{X}$.

Definition 3.1 *The risk measures ρ and $\rho_{\mathcal{G}}$ are called* acceptance (rejection) consistent *w. r. t. $\mathcal{Y}$ if the following condition holds for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$:*

$$\rho_{\mathcal{G}}(X) \ \leq (\geq)\ \rho_{\mathcal{G}}(Y) \quad \Rightarrow \quad \rho(X) \ \leq (\geq)\ \rho(Y). \tag{6}$$

As mentioned above, (6) reflects a backward approach to dynamic risk assessment. If the risk of a financial position $X \in \mathcal{X}$ is smaller (larger) than the risk of some test position $Y \in \mathcal{Y}$ under the conditional risk measure $\rho_{\mathcal{G}}$, then the same relation should hold under the unconditional risk measure ρ.

Remark 3.2 *Let $\mathcal{Y} \subset \mathcal{Y}' \subset \mathcal{X}$ be two test sets. If ρ and $\rho_{\mathcal{G}}$ are acceptance or rejection consistent w. r. t. $\mathcal{Y}'$, then they are also acceptance or rejection consistent w. r. t. $\mathcal{Y}$.*

(i) The smallest possible test set is the set of constant positions. Therefore, we speak of weak time consistency if $\mathcal{Y} = \mathbb{R}$.

(ii) The largest possible test set contains all financial positions. We thus use the notion of strong time consistency if $\mathcal{Y} = \mathcal{X}$. Note that in this case acceptance and rejection consistency have the same meaning.

The properties of time consistency imply certain structures of the acceptance and rejection sets. For instance, from the assumption $0 \in \mathcal{Y}$, we easily deduce that $\mathcal{A}_\mathcal{G} \subset \mathcal{A}$ if ρ and $\rho_\mathcal{G}$ are acceptance consistent. If the they are rejection consistent, then $\mathcal{N}_\mathcal{G} \subset \mathcal{N}$.

Strong time consistency has been analysed by, e. g., [2] and [15] in the coherent case, and by [5], [14] and [8] in the convex case. Weakly consistent risk measures have been investigated by [21] and [16]. For a generalisation of the notion of time consistency w. r. t. some test set to the situation of non-trivial initial information, we refer the interested reader to [19].

The following theorem provides a characterisation of acceptance consistency.

Theorem 3.3 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_\mathcal{G} : \mathcal{X} \to \mathcal{X}_\mathcal{G}$ be convex risk measures, and let $\mathcal{Y} \subset \mathcal{X}$ be a test set. The following conditions are equivalent:*

(a) ρ and $\rho_\mathcal{G}$ are acceptance consistent w. r. t. $\mathcal{Y}$.

(b) $\mathcal{A}_\mathcal{G} - \rho_\mathcal{G}(\mathcal{Y} \cap \mathcal{A}) \subset \mathcal{A}$.

Moreover, acceptance consistency of ρ and $\rho_\mathcal{G}$ w. r. t. $\mathcal{Y}$ implies the following equivalent conditions, where $\mathcal{X}^\mathcal{Y} := \{X \in \mathcal{X} \mid -\rho_\mathcal{G}(X) \in \mathcal{Y}\}$:

(c) $\rho(X) \leq \rho(-\rho_\mathcal{G}(X))$ for all $X \in \mathcal{X}^\mathcal{Y}$.

(d) $(\mathcal{A}_\mathcal{G} + \mathcal{Y} \cap \mathcal{X}_\mathcal{G} \cap \mathcal{A}) \cap \mathcal{X}^\mathcal{Y} \subset \mathcal{A}$.

(e) $\mathcal{N} \cap \mathcal{X}^\mathcal{Y} \subset \mathcal{N}_\mathcal{G} + \mathcal{Y} \cap \mathcal{X}_\mathcal{G} \cap \mathcal{N}$.

(f) For all $X \in \mathcal{X}^\mathcal{Y}$ and $Y \in \mathcal{Y} \cap \mathcal{X}_\mathcal{G}$ holds: $\rho_\mathcal{G}(X) \leq \rho_\mathcal{G}(Y) \Rightarrow \rho(X) \leq \rho(Y)$.

Proof: To show that (a) implies (b), we choose $X \in \mathcal{A}_\mathcal{G}$ and $Y \in \mathcal{Y} \cap \mathcal{A}$. Then $\rho_\mathcal{G}(X - \rho_\mathcal{G}(Y)) \leq \rho_\mathcal{G}(Y)$, and acceptance consistency yields $\rho(X - \rho_\mathcal{G}(Y)) \leq \rho(Y) \leq 0$.

To prove the other direction, we consider $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ such that $\rho_\mathcal{G}(X) \leq \rho_\mathcal{G}(Y)$. By translation invariance, $X + \rho_\mathcal{G}(X) \in \mathcal{A}_\mathcal{G}$ and $Y + \rho(Y) \in \mathcal{A}$. We have $X + \rho(Y) \geq X + \rho_\mathcal{G}(X) - \rho_\mathcal{G}(Y + \rho(Y))$. Monotonicity yields $X + \rho(Y) \in \mathcal{A}_\mathcal{G} - \rho_\mathcal{G}(\mathcal{Y} \cap \mathcal{A}) \subset \mathcal{A}$, and we obtain $\rho(X) - \rho(Y) \leq 0$ as desired.

Now, suppose that ρ and $\rho_\mathcal{G}$ are acceptance consistent and consider some $X \in \mathcal{X}^\mathcal{Y}$. Then $\rho_\mathcal{G}(X) = \rho_\mathcal{G}(-\rho_\mathcal{G}(X))$ and $-\rho_\mathcal{G}(X) \in \mathcal{Y}$. This yields (c).

We show equivalence of (c) and (d). If (c) holds, and if $X \in \mathcal{A}_\mathcal{G}$ and $Y \in \mathcal{Y} \cap \mathcal{X}_\mathcal{G} \cap \mathcal{A}$ are such that $X + Y \in \mathcal{X}^\mathcal{Y}$, then $\rho(X + Y) \leq \rho(-\rho_\mathcal{G}(X + Y)) = \rho(-\rho_\mathcal{G}(X) + Y) \leq \rho(Y) \leq 0$. Conversely, we have $X + \rho(-\rho_\mathcal{G}(X)) \in \mathcal{X}^\mathcal{Y}$ for $X \in \mathcal{X}^\mathcal{Y}$. This yields $\rho(X) \leq \rho(-\rho_\mathcal{G}(X))$ because $X + \rho(-\rho_\mathcal{G}(X)) = X + \rho_\mathcal{G}(X) - \rho_\mathcal{G}(X) + \rho(-\rho_\mathcal{G}(X)) \in \mathcal{A}_\mathcal{G} + \mathcal{Y} \cap \mathcal{X}_\mathcal{G} \cap \mathcal{A} \subset \mathcal{A}$. The proof of equivalence of (c) and (e) is similar and left to the reader.

Finally, we assume (c) and consider $X \in \mathcal{X}^\mathcal{Y}$ and $Y \in \mathcal{Y} \cap \mathcal{X}_\mathcal{G}$ with $\rho_\mathcal{G}(X) \leq \rho_\mathcal{G}(Y) = -Y$. Monotonicity yields (f) since $\rho(X) \leq \rho(-\rho_\mathcal{G}(X)) \leq \rho(Y)$. Conversely, (f) implies (c) because $-\rho_{\mathcal{G}}(X) \in \mathcal{Y} \cap \mathcal{X}_{\mathcal{G}}$ and $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}(-\rho_{\mathcal{G}}(X))$ for $X \in \mathcal{X}^{\mathcal{Y}}$. □

A characterisation of rejection consistency is given below. The proof is analogous.

Theorem 3.4 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be convex risk measures, and let $\mathcal{Y} \subset \mathcal{X}$ be a test set. The following conditions are equivalent:*

(a) ρ and $\rho_{\mathcal{G}}$ are rejection consistent w. r. t. $\mathcal{Y}$.

(b) $\mathcal{N}_{\mathcal{G}} - \rho_{\mathcal{G}}(\mathcal{Y} \cap \mathcal{N}) \subset \mathcal{N}$.

Moreover, rejection consistency of ρ and $\rho_{\mathcal{G}}$ w. r. t. $\mathcal{Y}$ implies the following equivalent conditions, where $\mathcal{X}^{\mathcal{Y}} := \{X \in \mathcal{X} \mid -\rho_{\mathcal{G}}(X) \in \mathcal{Y}\}$:

(c) $\rho(X) \geq \rho(-\rho_{\mathcal{G}}(X))$ for all $X \in \mathcal{X}^{\mathcal{Y}}$.

(d) $\mathcal{A} \cap \mathcal{X}^{\mathcal{Y}} \subset \mathcal{A}_{\mathcal{G}} + \mathcal{Y} \cap \mathcal{X}_{\mathcal{G}} \cap \mathcal{A}$.

(e) $(\mathcal{N}_{\mathcal{G}} + \mathcal{Y} \cap \mathcal{X}_{\mathcal{G}} \cap \mathcal{N}) \cap \mathcal{X}^{\mathcal{Y}} \subset \mathcal{N}$.

(f) For all $X \in \mathcal{X}^{\mathcal{Y}}$ and $Y \in \mathcal{Y} \cap \mathcal{X}_{\mathcal{G}}$ holds: $\rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y) \Rightarrow \rho(X) \geq \rho(Y)$.

Example: Let us consider the test set $\mathcal{Y} = \mathcal{X}_{\mathcal{G}}$ that contains all financial positions which are measurable w. r. t. the information that is available. We have $\mathcal{X} = \mathcal{X}^{\mathcal{Y}}$. Hence, conditions (a) and (b) are equivalent to (c)–(f) both in theorem 3.3 and 3.4.

The risk measures ρ and $\rho_{\mathcal{G}}$ are acceptance consistent w. r. t. $\mathcal{X}_{\mathcal{G}}$ iff $\mathcal{A}_{\mathcal{G}} + \mathcal{X}_{\mathcal{G}} \cap \mathcal{A} \subset \mathcal{A}$, or iff $\mathcal{N} \subset \mathcal{N}_{\mathcal{G}} + \mathcal{X}_{\mathcal{G}} \cap \mathcal{N}$, or iff $\rho(X) \leq \rho(-\rho_{\mathcal{G}}(X))$ for all $X \in \mathcal{X}$. They are rejection consistent iff the reversed inclusions and inequalities hold.

Moreover, the risk measures are acceptance *and* rejection consistent w. r. t. $\mathcal{X}_{\mathcal{G}}$ iff they are acceptance *or* rejection consistent w. r. t. $\mathcal{X}$, i. e. strongly time consistent. ◇

In section 6 and 7 we discuss whether these backward conditions of time consistency are appropriate to specify a reasonable update of a convex risk measure when new information arrives. We consider the strong and the weak version of acceptance consistency. Therefore, let us summarise the results of theorem 3.3 for the largest possible test set $\mathcal{Y} = \mathcal{X}$ and for the smallest possible test set $\mathcal{Y} = \mathbb{R}$.

Corollary 3.5 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be convex risk measures. The following conditions are equivalent:*

(a) ρ and $\rho_{\mathcal{G}}$ are strongly acceptance consistent.

(b) ρ and $\rho_{\mathcal{G}}$ are strongly rejection consistent.

(c) ρ and $\rho_{\mathcal{G}}$ are acceptance and rejection consistent w. r. t. $\mathcal{X}_{\mathcal{G}}$.

(d) $\rho(X) = \rho(-\rho_{\mathcal{G}}(X))$ for all $X \in \mathcal{X}$.

(e) $\mathcal{A} = \mathcal{A}_{\mathcal{G}} + \mathcal{X}_{\mathcal{G}} \cap \mathcal{A}$.

(f) $\mathcal{N} = \mathcal{N}_{\mathcal{G}} + \mathcal{X}_{\mathcal{G}} \cap \mathcal{N}$.

Corollary 3.6 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be convex risk measures. The following conditions are equivalent:*

(a) *ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent.*

(a') *For all $X \in \mathcal{X}$ holds: $\rho(X) \leq 0$ whenever $\rho_{\mathcal{G}}(X) \leq 0$.*

(b) $\mathcal{A}_{\mathcal{G}} \subset \mathcal{A}$.

(b') $\mathcal{A}_{\mathcal{G}} \subset \{X \in \mathcal{A} \mid X\mathbb{I}_A \in \mathcal{A} \;\forall\; A \in \mathcal{G}\}$.

Proof: (a) and (b) are equivalent by theorem 3.3. (a) and (a') are equivalent by translation invariance. (b') implies (b). Finally, consider some $X \in \mathcal{A}_{\mathcal{G}}$. We have $\rho_{\mathcal{G}}(X\mathbb{I}_A) = \rho_{\mathcal{G}}(X)\mathbb{I}_A$ for all $A \in \mathcal{G}$, cf. proposition 4.3 in [5]. Thus, $\rho_{\mathcal{G}}(X\mathbb{I}_A) \leq 0$. This means, $X\mathbb{I}_A \in \mathcal{A}_{\mathcal{G}} \subset \mathcal{A}$ for all $A \in \mathcal{G}$. □

4 Consecutivity

From now on, let us call the set

$$\mathcal{A}(\mathcal{G}) := \{X \in \mathcal{A} \mid X\mathbb{I}_A \in \mathcal{A} \;\forall\; A \in \mathcal{G}\} \tag{7}$$

appearing in condition (b') of the previous corollary the acceptance set of the convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ w. r. t. the additional information $\mathcal{G} \subset \mathcal{F}$.

Remark 4.1 *$\mathcal{A}(\mathcal{G})$ is non-empty, convex, and solid in the sense that $Y \in \mathcal{A}(\mathcal{G})$ whenever there is some $X \in \mathcal{A}(\mathcal{G})$ with $X \leq Y$. If ρ is coherent, $\mathcal{A}(\mathcal{G})$ is a convex cone. If ρ is sensitive as defined in (4), then $0 = \operatorname{ess\,inf}\; \mathcal{A}(\mathcal{G}) \cap \mathcal{X}_{\mathcal{G}}$. However, in general, $\mathcal{A}(\mathcal{G})$ is not conditionally convex and, thus, not an acceptance set of a conditional convex risk measure.*

Therewith we are able to introduce the notion of consecutivity which corresponds to an alternative approach to dynamic risk evaluation that is forward-directed in time. The risk of a financial position $X \in \mathcal{A}(\mathcal{G})$ is accepted by the initial risk measure ρ no matter which $\mathcal{G}$-measurable event occurs in the future. An update $\rho_{\mathcal{G}}$ of ρ should be compatible with previous evaluations and the incoming information $\mathcal{G}$. Thus, it makes sense that the conditional convex risk measure $\rho_{\mathcal{G}}$ should find the risk of X acceptable, too. This motivates the following definition.

Definition 4.2 *A conditional convex risk measure $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ is called a* consequence *of $\rho : \mathcal{X} \to \mathbb{R}$ and the incoming information $\mathcal{G} \subset \mathcal{F}$ if*

$$\mathcal{A}(\mathcal{G}) \subset \mathcal{A}_{\mathcal{G}}. \tag{8}$$

Consecutivity of two risk measures as defined in (8) can obviously be regarded as the forward-directed version of weak acceptance consistency, cf. corollary 3.6 above. It is equivalent to the following implication:

$$\rho(X\mathbb{I}_A) \leq 0 \quad \forall\; A \in \mathcal{G} \quad \Rightarrow \quad \rho_{\mathcal{G}}(X) \leq 0.$$

5 Robust shortfall risk measures

In this section we want to illustrate both approaches to dynamic risk assessment. We consider the class of robust shortfall risk measures. Let $l : \mathbb{R} \to \mathbb{R}$ be a convex and strictly increasing loss function and $\mathcal{Q}$ a convex set of probability measures which are equivalent to the reference measure P. The acceptance set

$$\mathcal{A} = \{X \in \mathcal{X} \mid \mathbb{E}_Q(l(-X)) \leq l(0) \;\forall\; Q \in \mathcal{Q}\} \tag{9}$$

generates as in (1) a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ that is continuous from below, continuous from above, and sensitive as in (4). Hence, ρ admits a robust representation in terms of equivalent probability measures. For further properties and for the specific structure of the minimal penalty function $\alpha_{\min}$, we refer to [11], section 4.6. Jensen's inequality yields, $\alpha_{\min}(Q) = 0$ for every probability measure $Q \in \mathcal{Q}$. Thus, a robust shortfall risk measure satisfies (5).

In the following proposition we characterise the acceptance set $\mathcal{A}(\mathcal{G})$ of ρ w. r. t. to new information $\mathcal{G} \subset \mathcal{F}$. Note that in this case $\mathcal{A}(\mathcal{G})$ is conditionally convex and may be regarded as an acceptance set of a conditional convex risk measure.

Proposition 5.1 *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a robust shortfall risk measure with an acceptance set as in (9). The acceptance set of ρ w. r. t. additional information $\mathcal{G} \subset \mathcal{F}$ is given by*

$$\mathcal{A}(\mathcal{G}) = \{X \in \mathcal{X} \mid \mathbb{E}_Q(l(-X)|\mathcal{G}) \leq l(0) \;\forall\; Q \in \mathcal{Q}\}.$$

Proof: We consider a financial position X that is contained in the set on the right-hand side with $\mathbb{E}_Q(l(-X)|\mathcal{G}) \leq l(0)$ for every $Q \in \mathcal{Q}$. This yields,

$$\mathbb{E}_Q(l(-X\mathbb{I}_A)) = \mathbb{E}_Q(\mathbb{E}_Q(l(-X)|\mathcal{G}); A) + l(0)Q(A^c) \leq l(0) \tag{10}$$

for all $A \in \mathcal{G}$. Hence, $X \in \mathcal{A}(\mathcal{G})$.

To show the other inclusion, we assume, there is some $X \in \mathcal{A}(\mathcal{G})$ and an equivalent probability measure $Q \in \mathcal{Q}$ such that the event $A := \{\mathbb{E}_Q(l(-X)|\mathcal{G}) > l(0)\}$ has strictly positive probability under P and Q. Since $A \in \mathcal{G}$ and $X\mathbb{I}_A \in \mathcal{A}$, we obtain as in (10) that $\mathbb{E}_Q(\mathbb{E}_Q(l(-X)|\mathcal{G}); A) \leq l(0) - l(0)Q(A^c) = l(0)Q(A)$. On the other hand, we have $\mathbb{E}_Q(\mathbb{E}_Q(l(-X)|\mathcal{G}); A) > l(0)Q(A)$, a contradiction. □

Let us define the corresponding conditional robust shortfall risk measure $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ where $\mathcal{G} \subset \mathcal{F}$ is a σ-field. We choose another convex and strictly increasing loss function $l_{\mathcal{G}} : \mathbb{R} \to \mathbb{R}$. For the sake of simplicity, we assume that the loss function is deterministic and does not depend on $\mathcal{G}$. We also fix a convex set $\mathcal{Q}_{\mathcal{G}}$ of equivalent probability measures. The conditional acceptance set

$$\mathcal{A}_{\mathcal{G}} = \{X \in \mathcal{X} \mid \mathbb{E}_Q(l_{\mathcal{G}}(-X)|\mathcal{G}) \leq l_{\mathcal{G}}(0) \;\forall\; Q \in \mathcal{Q}_{\mathcal{G}}\} \tag{11}$$

generates as in (1) a conditional convex risk measure which has similar properties as the unconditional robust shortfall risk measure.

Corollary 3.6 and proposition 5.1 yield the following result.

Corollary 5.2 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be robust shortfall risk measures with acceptance sets as in (9) and (11). If*

$$l = l_{\mathcal{G}} \quad \text{and} \quad \mathcal{Q} = \mathcal{Q}_{\mathcal{G}},$$

then ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent, and $\rho_{\mathcal{G}}$ is a consequence of ρ and $\mathcal{G}$.

Remark 5.3 *Let us consider the non-robust case where the acceptance sets of ρ and $\rho_{\mathcal{G}}$ are as in (9) and (11) with $l = l_{\mathcal{G}}$ and $\mathcal{Q} = \mathcal{Q}_{\mathcal{G}} = \{Q\}$. By proposition 5.1 and corollary 5.2, we have $\mathcal{A}(\mathcal{G}) = \mathcal{A}_{\mathcal{G}}$ which yields that ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent, and $\rho_{\mathcal{G}}$ is a consequence of ρ and $\mathcal{G}$. In [21], theorem 4.15, is shown that these risk measures are also weakly rejection consistent. However, strong time consistency may fail as illustrated by example 3.5 in [17].*

The set of equivalent probability measures reflects our uncertainty about the underlying probabilistic model which may decrease or increase when new information arrives.

Corollary 5.4 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be robust shortfall risk measures with acceptance sets as in (9) and (11), and suppose that $l = l_{\mathcal{G}}$. Then:*

(i) ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent if $\mathcal{Q} \subset \mathcal{Q}_{\mathcal{G}}$.

(ii) $\rho_{\mathcal{G}}$ is a consequence of ρ and the incoming information $\mathcal{G}$ if $\mathcal{Q}_{\mathcal{G}} \subset \mathcal{Q}$.

The loss function reflects our attitude towards risk. Here, too, additional information may cause some change. Consider, for instance, the Arrow-Pratt coefficient of absolute risk aversion defined as

$$\alpha(y) := \frac{l''(y)}{l'(y)} \geq 0 \quad \text{and} \quad \alpha_{\mathcal{G}}(y) := \frac{l''_{\mathcal{G}}(y)}{l'_{\mathcal{G}}(y)} \geq 0$$

for twice continuously differentiable loss functions l and $l_{\mathcal{G}}$.

Corollary 5.5 *Let $\rho : \mathcal{X} \to \mathbb{R}$ and $\rho_{\mathcal{G}} : \mathcal{X} \to \mathcal{X}_{\mathcal{G}}$ be robust shortfall risk measures with acceptance sets as in (9) and (11). Suppose that $\mathcal{Q} = \mathcal{Q}_{\mathcal{G}}$, and that the loss functions l and $l_{\mathcal{G}}$ are twice continuously differentiable. Then:*

(i) ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent if $\alpha \leq \alpha_{\mathcal{G}}$.

(ii) $\rho_{\mathcal{G}}$ is a consequence of ρ and the incoming information $\mathcal{G}$ if $\alpha_{\mathcal{G}} \leq \alpha$.

Proof: We only show the first part, the proof of the second is analogous. We assume that $\alpha(y) \leq \alpha_{\mathcal{G}}(y)$ for all $y \in \mathbb{R}$, and choose some $X \in \mathcal{A}_{\mathcal{G}}$. There exists a strictly increasing and convex function F such that $l_{\mathcal{G}} = F \circ l$, cf. proposition 2.47 in [11]. Jensen's inequality yields,

$$F(\mathbb{E}_Q(l(-X)|\mathcal{G})) \leq \mathbb{E}_Q(l_{\mathcal{G}}(-X)|\mathcal{G}) \leq l_{\mathcal{G}}(0) \quad \forall\, Q \in \mathcal{Q}.$$

Thus,

$$\mathbb{E}_Q(l(-X)|\mathcal{G}) \leq F^{-1}(l_{\mathcal{G}}(0)) = l(0) = l(0) \quad \forall\, Q \in \mathcal{Q}.$$

Hence, $X \in \mathcal{A}(\mathcal{G})$, which means that ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent. □

6 Strong time consistency as an update criterion

In this section we analyse whether strong time consistency, cf. corollary 3.5, is an appropriate update criterion for convex risk measures. Given some $\rho : \mathcal{X} \to \mathbb{R}$, does an update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ exist such that, for every σ-field, ρ and the associated conditional convex risk measure $\rho_{\mathcal{G}}$ are strongly time consistent?

Example: There are only three known types of risk measures which can be updated under the criterion of strong time consistency: the worst-case risk measure, linear and entropic risk measures.

(i) The worst-case risk measure $\rho^{\text{worst}} : \mathcal{X} \to \mathbb{R}$ is defined as

$$\rho^{\text{worst}}(X) := \inf\{m \in \mathbb{R} \mid X + m \geq 0\} \quad \forall\, X \in \mathcal{X}.$$

Let $\mathcal{G} \mapsto \rho^{\text{worst}}_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ be the update rule that assigns to every σ-field the corresponding conditional worst-case risk measure given by

$$\rho^{\text{worst}}_{\mathcal{G}}(X) := \operatorname{ess\,inf}\{X' \in \mathcal{X}_{\mathcal{G}} \mid X + X' \geq 0\} \quad \forall\, X \in \mathcal{X}.$$

We have $\mathcal{A}^{\text{worst}} = \mathcal{A}^{\text{worst}}_{\mathcal{G}} = \mathcal{X}_+$ for all $\mathcal{G} \subset \mathcal{F}$. By corollary 3.5 (e), ρ^{worst} and $\rho^{\text{worst}}_{\mathcal{G}}$ are always strongly time consistent.

(ii) Let $\rho : \mathcal{X} \to \mathbb{R}$ be a linear risk measure that is sensitive as in (4), i. e., there is some equivalent probability measure $Q \sim P$ such that

$$\rho(X) = \mathbb{E}_Q(-X) \quad \forall\, X \in \mathcal{X}.$$

The usual conditional expectation provides an update of ρ when new information arrives. We consider the update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ with

$$\rho_{\mathcal{G}}(X) := \mathbb{E}_Q(-X|\mathcal{G}) \quad \forall\, X \in \mathcal{X}.$$

By corollary 3.5 (d), ρ and $\rho_{\mathcal{G}}$ are always strongly time consistent.

(iii) We consider an entropic risk measure $\rho : \mathcal{X} \to \mathbb{R}$ that is sensitive as in (4), i. e., there are a parameter $\beta > 0$ and an equivalent probability measure $Q \sim P$ such that

$$\rho(X) := \frac{1}{\beta} \log \mathbb{E}_Q(e^{-\beta X}) \quad \forall\, X \in \mathcal{X}.$$

Let $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ be the update rule that assigns to every σ-field the corresponding conditional entropic risk measure

$$\rho_{\mathcal{G}}(X) := \frac{1}{\beta} \log \mathbb{E}_Q(e^{-\beta X}|\mathcal{G}) \quad \forall\, X \in \mathcal{X}.$$

By corollary 3.5 (d), ρ and $\rho_{\mathcal{G}}$ are always strongly time consistent.

$\diamond$

We now characterise those risk measure $\rho : \mathcal{X} \to \mathbb{R}$ which admit a stronly time consistent update rule. We assume that ρ has a robust representation in terms of equivalent probability measures as in (3) and (5).

Remark 6.1 *We fix a σ-field $\mathcal{G} \subset \mathcal{F}$ and assume, there is some $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ such that ρ and $\rho_{\mathcal{G}}$ are strongly time consistent. In particular, ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent, i. e. $\mathcal{A}_{\mathcal{G}} \subset \mathcal{A}(\mathcal{G})$. In fact, strong time consistency and sensitivity of ρ yield*

$$\mathcal{A}_{\mathcal{G}} = \mathcal{A}(\mathcal{G}),$$

as shown by [3] in proposition 4.7. If there was a financial position $X \in \mathcal{A}(\mathcal{G})$ such that the event $A := \{\rho_{\mathcal{G}}(X) > 0\} \in \mathcal{G}$ had strictly positive probability under P, then one would obtain the following contradiction:

$$0 \geq \rho(X\mathbb{I}_A) = \rho(-\rho_{\mathcal{G}}(X\mathbb{I}_A)) = \rho(-\rho_{\mathcal{G}}(X)\mathbb{I}_A) > 0.$$

Hence, there exists at most one conditional convex risk measure such that strong time consistency holds.

Another characterisation is provided by the following theorem. Under some slightly stronger notion of sensitivity

$$P(A) > 0 \quad \Rightarrow \quad \rho(x\mathbb{I}_A) > \rho(y\mathbb{I}_A) \quad \forall\, x < y \leq 0, \tag{12}$$

we show that existence of a strongly time consistent update rule yields that ρ is uniquely determined by its values on simple functions of the form $x\mathbb{I}_A$. This indicates that strong time consistency may fail as a general update criterion.

Note that (12) is equivalent to (4) if ρ is positively homogeneous. Hence, every sensitive coherent risk measure is also strongly sensitive.

Theorem 6.2 *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure that satisfies (5) and (12). If there exists an update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ such that ρ and $\rho_{\mathcal{G}}$ are always strongly time consistent, then ρ is uniquely determined by its values on the class of all simple functions of the form $x\mathbb{I}_A$ where $x \in (-\infty, 0]$ and $A \in \mathcal{F}$.*

Proof: For a fixed $A \in \mathcal{F}$ with $P(A) > 0$, we consider the mapping $x \mapsto \rho(x\mathbb{I}_A)$. It is convex, Lipschitz continuous, and strictly decreasing on $(-\infty, 0]$ by strong sensitivity. Moreover, $|x|\rho(-\mathbb{I}_A) \leq \rho(x\mathbb{I}_A)$ for $x \leq -1$. Hence, $\rho(x\mathbb{I}_A) \to \infty$ as $x \to -\infty$. This yields, for every $y \geq 0$, there is a unique solution $x \leq 0$ to the equation

$$\rho(x\mathbb{I}_A) = y. \tag{13}$$

Now, we choose some $X \in \mathcal{X}$. We set $\mathcal{G} = \sigma(A)$, and the update rule assigns a conditional convex risk measure $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$. We denote by $\rho_A(X)$ the constant value of $\rho_{\mathcal{G}}(X)$ on A. Due to strong time consistency, $\rho_A(X)$ solves the equation $\rho(X\mathbb{I}_A) = \rho(-x\mathbb{I}_A)$ with $x \in \mathbb{R}$. If $X \leq 0$, then the solution is unique.

To prove that ρ is uniquely determined by its values on simple functions of the form $x\mathbb{I}_A$, we show, for every step function, $\rho(X)$ can be calculated recursively by solving an equation of type (13). Any other $X \in \mathcal{X}$ can be approximated from above by step functions, and the assertion follows by continuity from above.

Let X be a simple function, i. e., there are a finite partition $\{A_i\}_{i \leq n} \subset \mathcal{F}$ and $x_i \in \mathbb{R}$ so that $X = \sum_1^n x_i \mathbb{I}_{A_i}$. By translation invariance, we assume w. l. o. g. that $x_i \leq x_n = 0$

for all $i < n$. For $k \leq n-1$, we set $X^k := \sum_1^k x_i \mathbb{I}_{A_i}$ and $y_k := \rho(X^k) \geq 0$. Obviously, $X^{n-1} = X$ and $y_{n-1} = \rho(X)$. For all $k \leq n-1$, there is a unique solution $x \leq 0$ to the equation $\rho(x\mathbb{I}_{A^c_{k+1}}) = y_k$, and strong time consistency w.r.t. $\mathcal{G} := \sigma(A_{k+1})$ yields, $x = -\rho_{A^c_{k+1}}(X^{k+1})$. Hence, $y_{k+1} = \rho(-\rho_{A^c_{k+1}}(X^{k+1})\mathbb{I}_{A^c_{k+1}} + x_{k+1}\mathbb{I}_{A_{k+1}}) = \rho((x - x_{k+1})\mathbb{I}_{A^c_{k+1}}) - x_{k+1}$. Further details of the proof are left to the reader. □

To finish this section, we discuss the updating problem for the class of coherent risk measures which satisfy the following condition:

$$P(A) > 0, \quad A \in \mathcal{F} \quad \Rightarrow \quad \rho(\mathbb{I}_A) < 0. \tag{14}$$

We show that it is not possible to update these risk measures under the criterion of strong time consistency except for the trivial linear case. Note that strong sensitivity of ρ is implied by (14) because for all $A \in \mathcal{F}$ with $P(A) > 0$ holds $0 < -\rho(\mathbb{I}_A) \leq \rho(-\mathbb{I}_A)$.

The theorem below was motivated by [22]. There is shown on finite spaces that the Choquet integral w.r.t. a monotone set function $c : \mathcal{F} \to [0,1]$ satisfies the "Iterative law of expectation" if and only if c is a probability measure, i.e.,

$$\mathbb{E}_c(\mathbb{E}_c(X|\mathcal{G})) = \mathbb{E}_c(X) \quad \forall\, X \in \mathcal{X},\ \mathcal{G} \subset \mathcal{F} \quad \Leftrightarrow \quad c \in \mathcal{M}_1.$$

This means, a non-linear comonotone risk measure $\rho(X) := -\mathbb{E}_c(X)$ does not admit a strongly time consistent update rule $\mathcal{G} \mapsto \rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}$ with $\rho_\mathcal{G}(X) := -\mathbb{E}_c(X|\mathcal{G})$. We drop the condition of comonotonicity and the assumption of a finite space.

Theorem 6.3 *Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a coherent risk measure that satisfies (14). Let $\mathcal{G} \mapsto \rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}$ be an update rule for ρ. Then the following conditions are equivalent:*

(a) For every σ-field $\mathcal{G} \subset \mathcal{F}$, ρ and $\rho_\mathcal{G}$ are strongly time consistent.

(b) ρ is linear, and every update of ρ is given by the conditional expectation.

Proof: Let $\mathcal{G} \mapsto \rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}$ be a strongly time consistent update rule for ρ. In the coherent case, (5) is equivalent to sensitivity as in (4) and strong sensitivity as in (12). Moreover, these conditions are implied by (14). By theorem 6.2 and positive homogeneity, we have to show that the set function given by $Q(A) := \rho(-\mathbb{I}_A)$, $A \in \mathcal{F}$, is a probability measure on $(\Omega, \mathcal{F})$ and equivalent to P. Q is a monotone set function, normalised, continuous from below and subadditive. By sensitivity of ρ, we have $Q(A) = 0$ iff $P(A) = 0$. All we have to show is superadditivity of Q or

$$\rho(-\mathbb{I}_A) = -\rho(\mathbb{I}_A) \quad \forall\, A \in \mathcal{F}. \tag{15}$$

To prove (15), we fix some $A \in \mathcal{F}$ and consider the σ-field $\mathcal{G} := \sigma(A)$. We assume w.l.o.g. that $0 < P(A) < 1$. Otherwise (15) is obvious. The update rule assigns a conditional risk measure $\rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}$. For every $X \in \mathcal{X}$, the random variable $\rho_\mathcal{G}(X)$ is constant on A, and we denote this value by $\rho_A(X)$. Strong time consistency and positive homogeneity yield $\rho(X\mathbb{I}_A) = \rho(-\rho_A(X)\mathbb{I}_A) = \rho_A(X)\rho(-\mathbb{I}_A)$ for all $0 \geq X \in \mathcal{X}$. Since $\rho(-\mathbb{I}_A) > 0$, we have

$$\rho_A(X) = \frac{\rho(X\mathbb{I}_A)}{\rho(-\mathbb{I}_A)} \quad \forall\, 0 \geq X \in \mathcal{X}. \tag{16}$$

By (14), we can choose a partition $\Omega = A_1 \cup A_2 \cup A_3$ with $\rho(\mathbb{I}_{A_i}) < 0$ for all $i \leq 3$ such that $A \in \sigma(A_i, i \leq 3) =: \mathcal{G}$. For $i \neq j$, we define $X^{ij}_\beta := -\mathbb{I}_{A_i} - \beta\mathbb{I}_{A_j}$ where $\beta > 1$ is sufficiently large so that positive homogeneity of ρ may be applied in the equations below. Using strong time consistency w. r. t. $\mathcal{G} := \sigma(A_i)$ and (16), we obtain

$$\begin{aligned}
\rho(X^{ij}_\beta) &= \rho(-\rho_{A_i}(X^{ij}_\beta)\mathbb{I}_{A_i} - \rho_{A_i^c}(X^{ij}_\beta)\mathbb{I}_{A_i^c}) \\
&= \rho\left(-\mathbb{I}_{A_i} - \mathbb{I}_{A_i^c}\frac{\beta\rho(-\mathbb{I}_{A_j})}{\rho(-\mathbb{I}_{A_i^c})}\right) \\
&= \rho\left(\left[\frac{\beta\rho(-\mathbb{I}_{A_j})}{\rho(-\mathbb{I}_{A_i^c})} - 1\right]\mathbb{I}_{A_i}\right) + \frac{\beta\rho(-\mathbb{I}_{A_j})}{\rho(-\mathbb{I}_{A_i^c})} \\
&= -\rho(\mathbb{I}_{A_i}) + \beta\rho(-\mathbb{I}_{A_j})\left[\frac{\rho(\mathbb{I}_{A_i}) + 1}{\rho(-\mathbb{I}_{A_i^c})}\right] \\
&= -\rho(\mathbb{I}_{A_i}) + \beta\rho(-\mathbb{I}_{A_j}).
\end{aligned}$$

On the other hand, strong time consistency w. r. t. $\mathcal{G} := \sigma(A_j)$ yields

$$\begin{aligned}
\rho(X^{ij}_\beta) &= \rho(-\rho_{A_j}(X^{ij}_\beta)\mathbb{I}_{A_j} - \rho_{A_j^c}(X^{ij}_\beta)\mathbb{I}_{A_j^c}) \\
&= \rho\left(-\beta\mathbb{I}_{A_j} - \mathbb{I}_{A_j^c}\frac{\rho(-\mathbb{I}_{A_i})}{\rho(-\mathbb{I}_{A_j^c})}\right) \\
&= \rho\left(-\left[\beta - \frac{\rho(-\mathbb{I}_{A_i})}{\rho(-\mathbb{I}_{A_j^c})}\right]\mathbb{I}_{A_j}\right) + \frac{\rho(-\mathbb{I}_{A_i})}{\rho(-\mathbb{I}_{A_j^c})} \\
&= -\rho(-\mathbb{I}_{A_i})\left[\frac{\rho(-\mathbb{I}_{A_j}) - 1}{\rho(-\mathbb{I}_{A_j^c})}\right] + \beta\rho(-\mathbb{I}_{A_j}) \\
&= -\rho(-\mathbb{I}_{A_i})\frac{\rho(\mathbb{I}_{A_j^c})}{\rho(-\mathbb{I}_{A_j^c})} + \beta\rho(-\mathbb{I}_{A_j}).
\end{aligned}$$

Hence,

$$\frac{\rho(\mathbb{I}_{A_i})}{\rho(-\mathbb{I}_{A_i})} = \frac{\rho(\mathbb{I}_{A_j^c})}{\rho(-\mathbb{I}_{A_j^c})}.$$

Since i and j were chosen arbitrarily, this yields

$$0 > \frac{\rho(\mathbb{I}_B)}{\rho(-\mathbb{I}_B)} = K \geq -1 \quad \forall\, B \in \mathcal{G}\backslash\{\emptyset, \Omega\}. \tag{17}$$

In particular, the ratio equals K for the fixed event $A \in \mathcal{F}$. It remains to show, $K = -1$.

The restriction of ρ to the class of all bounded $\mathcal{G}$-measurable random variables is still a coherent risk measure satisfying (12). It admits a robust representation in terms of equivalent probability measures, i. e.

$$\rho(X) = \sup_{\mathcal{Q}} \mathbb{E}_Q(-X) \quad \forall\, X \in L^\infty(\Omega, \mathcal{G}, P)$$

where $\mathcal{Q}$ is some convex set of the form $\mathcal{Q} = \{(p, q, 1-p-q) \mid p \in [a, b],\ q \in [c, d]\}$.

(14) yields $0 < a \leq b$ and $0 < c \leq d$. We set $e := 1-b-d$, $f := 1-a-c$, and again we obtain $0 < e \leq f$. The constant ratio in (17) for $B = A_i$ and every $i \leq 3$ yields $a = -Kb$, $c = -Kd$, and $e = -Kf$. Hence, $(1-K^2)(b+d) = 1+K$.

Now, apply (17) to the event $B = A_2 \cup A_3$. Then $(1-K^2)b = 1+K$. If $K > -1$, then $b + d = b$, a contradiction to (14). This completes the proof of (b)⇒(a). The other direction is trivial. □

7 Weak acceptance consistency as an update criterion

In this section we use weak acceptance consistency as an update criterion, cf. corollary 3.6. The following example shows, there exists a weakly acceptance consistent update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ for every convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$. In general, weak acceptance consistency does not characterise an update rule uniquely.

Example:

(i) We consider the trivial worst-case update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}}^{\text{worst}} \in \mathcal{R}_{\mathcal{G}}$ with

$$\rho_{\mathcal{G}}^{\text{worst}}(X) := \operatorname{ess\,inf}\{X' \in \mathcal{X}_{\mathcal{G}} \mid X + X' \geq 0\} \quad \forall\, X \in \mathcal{X}.$$

If $\rho_{\mathcal{G}}^{\text{worst}}(X) \leq m \in \mathbb{R}$, then $X \geq -m$. Monotonicity and translation invariance yield, $\rho(X) \leq \rho(-m) = m$. Hence, ρ and $\rho_{\mathcal{G}}^{\text{worst}}$ are weakly acceptance consistent.

(ii) Suppose that ρ is a convex risk measure and let

$$\rho(X) = \sup\{\mathbb{E}_Q(-X) - \alpha_{\min}(Q) \mid Q \sim P\} \quad \forall\, X \in \mathcal{X}$$

be its robust representation. We can define the update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ with

$$\rho_{\mathcal{G}}(X) := \operatorname{ess\,sup}\{\mathbb{E}_Q(-X|\mathcal{G}) - \alpha_{\min}(Q) \mid Q \sim P\} \quad \forall\, X \in \mathcal{X}.$$

If $\rho_{\mathcal{G}}(X) \leq m$, then $\mathbb{E}_Q(-X|\mathcal{G}) - \alpha_{\min}(Q) \leq m$ for all $Q \sim P$. In particular, we have $\mathbb{E}_Q(-X) - \alpha_{\min}(Q) \leq m$ for all $Q \sim P$. Thus, $\rho(X) \leq m$, which yields weak acceptance consistency of ρ and $\rho_{\mathcal{G}}$.

◇

Now, we try to find the most tolerant weakly acceptance consistent update rule. For a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ and a σ-field $\mathcal{G} \subset \mathcal{F}$, let us denote by $\mathcal{R}_{\mathcal{G}}^{\leftarrow} \subset \mathcal{R}_{\mathcal{G}}$ the class of all conditional convex risk measures $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ such that ρ and $\rho_{\mathcal{G}}$ are weakly acceptance consistent. The left arrow indicates that the condition of weak acceptance consistency corresponds to a backward approach to dynamic risk assessment.

$\mathcal{R}_{\mathcal{G}}^{\leftarrow}$ is directed downwards w. r. t. the partial order (2):

$$\rho'_{\mathcal{G}} \preceq \rho_{\mathcal{G}} \quad \text{and} \quad \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}^{\leftarrow} \quad \Rightarrow \quad \rho'_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}^{\leftarrow}.$$

The conditional worst-case risk measure is least tolerant in $\mathcal{R}_{\mathcal{G}}^{\leftarrow}$. However, the question is whether it is possible to define an update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}}^{\star} \in \mathcal{R}_{\mathcal{G}}^{\leftarrow}$ that always assigns the most tolerant risk measure of this class.

If $\rho_\mathcal{G}^\star \in \mathcal{R}_\mathcal{G}$ is a conditional convex risk measure whose acceptance set is given by the acceptance set of ρ w. r. t. $\mathcal{G}$

$$\mathcal{A}_\mathcal{G}^\star = \mathcal{A}(\mathcal{G}), \tag{18}$$

then ρ and $\rho_\mathcal{G}^\star$ are weakly acceptance consistent, and $\rho_\mathcal{G}^\star$ is most tolerant in $\mathcal{R}_\mathcal{G}^{\leftarrow}$ by corollary 3.6. We already know that (18) applies to the class of robust shortfall risk measures, cf. proposition 5.1. For a general convex risk measure, we show that this condition is necessary for existence of a most tolerant risk measure in $\mathcal{R}_\mathcal{G}^{\leftarrow}$.

Theorem 7.1 *Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure that satisfies (5), and let $\mathcal{G} \subset \mathcal{F}$ be a σ-field. The acceptance set of ρ w. r. t. $\mathcal{G}$ has the following representation:*

$$\mathcal{A}(\mathcal{G}) = \bigcup_{\rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}^{\leftarrow}} \mathcal{A}_\mathcal{G}.$$

In particular, a conditional convex risk measure $\rho_\mathcal{G}^\star \in \mathcal{R}_\mathcal{G}$ is most tolerant in $\mathcal{R}_\mathcal{G}^{\leftarrow}$ if and only if its acceptance set $\mathcal{A}_\mathcal{G}^\star$ equals $\mathcal{A}(\mathcal{G})$.

Proof: By corollary 3.6, the union of all acceptance sets $\mathcal{A}_\mathcal{G}$ of conditional risk measures $\rho_\mathcal{G} \in \mathcal{R}_\mathcal{G}^{\leftarrow}$ is contained in $\mathcal{A}(\mathcal{G})$. To prove the other inclusion, we fix a $X \in \mathcal{A}(\mathcal{G})$. We show that there is a risk measure $\rho_\mathcal{G}^X \in \mathcal{R}_\mathcal{G}^{\leftarrow}$ with $\rho_\mathcal{G}^X(X) \leq 0$. Since ρ admits a robust representation,

$$\rho(Y) = \sup\{\mathbb{E}_Q(-Y) - \alpha_{\min}(Q) \mid Q \sim P\} \quad \forall\, Y \in \mathcal{X},$$

we have $Y \in \mathcal{A}(\mathcal{G})$ iff

$$\mathbb{E}_Q(-Y\mathbb{I}_A) \leq \alpha_{\min}(Q) \quad \forall\, Q \sim P,\ A \in \mathcal{G}. \tag{19}$$

We define $\rho_\mathcal{G}^X(Y) := \operatorname{ess\,sup}\left\{\mathbb{E}_Q(-Y|\mathcal{G}) - \alpha_\mathcal{G}^X(Q) \mid Q \sim P\right\}$, $Y \in \mathcal{X}$, where the penalty function is given by $\alpha_\mathcal{G}^X(Q) := \mathbb{E}_Q(-X|\mathcal{G})\mathbb{I}_{\{\mathbb{E}_Q(-X|\mathcal{G})\geq 0\}}$, $Q \sim P$. Obviously, $\rho_\mathcal{G}^X(X) \leq 0$. Moreover, $\rho_\mathcal{G}^X$ has the representation of a conditional convex risk measure. We only have to show normalisation. We choose some $Q^\star \sim P$ with $\alpha_{\min}(Q^\star) = 0$. Such an equivalent probability measure exists due to (5). By condition (19), we obtain

$$0 \geq \mathbb{E}_{Q^\star}(-X\mathbb{I}_{\{\mathbb{E}_{Q^\star}(-X|\mathcal{G})\geq 0\}}) = \mathbb{E}_{Q^\star}(\mathbb{E}_{Q^\star}(-X|\mathcal{G})\mathbb{I}_{\{\mathbb{E}_{Q^\star}(-X|\mathcal{G})\geq 0\}}) \geq 0,$$

which yields that $\mathbb{E}_{Q^\star}(-X|\mathcal{G}) \leq 0$. Hence, $0 \geq \rho_\mathcal{G}^X(0) \geq -\mathbb{E}_{Q^\star}(-X|\mathcal{G})\mathbb{I}_{\{\mathbb{E}_{Q^\star}(-X|\mathcal{G})\geq 0\}} = 0$.

To prove weak acceptance consistency or $\mathcal{A}_\mathcal{G}^X \subset \mathcal{A}(\mathcal{G})$, respectively, we choose some $Y \in \mathcal{A}_\mathcal{G}^X$. For all $Q \sim P$ holds $\mathbb{E}_Q(-Y|\mathcal{G}) \leq \mathbb{E}_Q(-X|\mathcal{G})\mathbb{I}_{\{\mathbb{E}_Q(-X|\mathcal{G})\geq 0\}}$ which yields,

$$\begin{aligned}
\mathbb{E}_Q(-Y\mathbb{I}_A) &= \mathbb{E}_Q(\mathbb{E}_Q(-Y|\mathcal{G})\mathbb{I}_A) \\
&\leq \mathbb{E}_Q(\mathbb{E}_Q(-X|\mathcal{G})\mathbb{I}_{\{\mathbb{E}_Q(-X|\mathcal{G})\geq 0\}}\mathbb{I}_A) \\
&= \mathbb{E}_Q(-X\mathbb{I}_{\{\mathbb{E}_Q(-X|\mathcal{G})\geq 0\}}\mathbb{I}_A) \\
&\leq \alpha_{\min}(Q).
\end{aligned}$$

for every event $A \in \mathcal{G}$. The assertion follows from (19). □

8 Consecutivity as an update criterion

Last but not least, we analyse to what extent the forward condition of consecutivity can be used for updating. Recall that under this criterion every update $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ has to be chosen as a consequence of the initial risk measure $\rho : \mathcal{X} \to \mathbb{R}$ and the incoming information $\mathcal{G} \subset \mathcal{F}$ in the sense that $\mathcal{A}(\mathcal{G}) \subset \mathcal{A}_{\mathcal{G}}$.

In general, the acceptance set $\mathcal{A}(\mathcal{G})$ of ρ w. r. t. $\mathcal{G}$ is a subset of $\mathcal{A}$. We do not require that any financial position accepted by ρ should be accepted by the new risk measure $\rho_{\mathcal{G}}$. The condition $\mathcal{A} \subset \mathcal{A}_{\mathcal{G}}$ would in fact be far too restrictive as the following remark shows.

Remark 8.1 *Let ρ be a convex risk measure, and suppose that there is an update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ such that $\mathcal{A} \subset \mathcal{A}_{\mathcal{G}}$ for every σ-field. Then we obtain for all $\mathcal{G} \in \mathcal{F}$,*

$$\operatorname{ess\,inf}\ \mathcal{X}_{\mathcal{G}} \cap \mathcal{A} \geq \operatorname{ess\,inf}\ \mathcal{X}_{\mathcal{G}} \cap \mathcal{A}_{\mathcal{G}} = 0.$$

This yields, $\mathcal{A} = \mathcal{X}_+$. Thus, the acceptance set $\mathcal{A}$ corresponds to the worst-case risk measure, i. e., $\rho = \rho^{\text{worst}}$.

For a convex risk measure that admits a robust representation in terms of equivalent probability measures, there exists at least one consequence provided that there is a probabilistic model which is not penalised under the minimal penalty function.

Example: Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure that satisfies (5). We choose a probability measure $Q^\star \sim P$ with $\alpha_{\min}(Q^\star) = 0$ and define the following update rule $\mathcal{G} \mapsto \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$. We set

$$\rho_{\mathcal{G}}(X) := \mathbb{E}_{Q^\star}(-X|\mathcal{G}) \quad \forall\ X \in \mathcal{X}$$

if the σ-field is non-trivial. Otherwise, $\rho_{\mathcal{G}}$ is given by ρ. As soon as we get any information, we are convinced that $Q^\star$ is the true probability measure which has to be used for risk assessment. We leave it to the reader to verify that every conditional linear risk measure $\rho_{\mathcal{G}}$ is a consequence of ρ and $\mathcal{G}$. Moreover, by linearity, every $\rho_{\mathcal{G}}$ can be interpreted as a maximal consequence, cf. remark 2.4. ◇

Consecutivity does not characterise an update rule uniquely. For a convex risk measure $\rho : X \to \mathbb{R}$ and a σ-field $\mathcal{G}$, we denote by $\mathcal{R}_{\mathcal{G}}^{\rightarrow} \subset \mathcal{R}_{\mathcal{G}}$ the class of all conditional risk measures $\rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}$ which are a consequence of ρ and $\mathcal{G}$ in the sense of definition 4.2. The right arrow indicates that the criterion corresponds to a forward approach to dynamic risk assessment.

$\mathcal{R}_{\mathcal{G}}^{\rightarrow}$ is directed upwards with regard to the partial order (2):

$$\rho'_{\mathcal{G}} \succeq \rho_{\mathcal{G}} \quad \text{and} \quad \rho_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}^{\rightarrow} \quad \Rightarrow \quad \rho'_{\mathcal{G}} \in \mathcal{R}_{\mathcal{G}}^{\rightarrow}.$$

The previous example shows that there is at least one maximal consequence. In general, a most tolerant consequence does not exist.

Remark 8.2 *Let ρ be a convex risk measure that satisfies (5), and let $\mathcal{G} \subset \mathcal{F}$ be a σ-field.*

(i) *If there is a weakly acceptance consistent risk measure $\rho^\star_{\mathcal{G}}$ that is most tolerant in $\mathcal{R}^{\leftarrow}_{\mathcal{G}}$, then $\mathcal{A}^\star_{\mathcal{G}} = \mathcal{A}(\mathcal{G})$ by theorem 7.1. On the other hand, $\rho^\star_{\mathcal{G}}$ is also the least tolerant consequence in $\mathcal{R}^{\rightarrow}_{\mathcal{G}}$.*

(ii) *If ρ is linear, i. e., $\rho(X) = \mathbb{E}_Q(-X)$ for all $X \in \mathcal{X}$ with some $Q \sim P$, then we have*

$$\mathcal{A}(\mathcal{G}) = \{X \in \mathcal{X} \mid \mathbb{E}_Q(-X|\mathcal{G}) \leq 0\}$$

by proposition 5.1. The acceptance set of ρ w. r. t. $\mathcal{G}$ equals the acceptance set of the conditional linear risk measure

$$\rho_{\mathcal{G}}(X) := \mathbb{E}_Q(-X|\mathcal{G}) \quad \forall\ X \in \mathcal{X}.$$

$\rho_{\mathcal{G}}$ is most tolerant in $\mathcal{R}^{\leftarrow}_{\mathcal{G}}$ and least tolerant in $\mathcal{R}^{\rightarrow}_{\mathcal{G}}$. In particular, the least consequence is also maximal, cf. remark 2.4. We can conclude, for linear risk measures, there exists a unique consequence which is given by the conditional expectation.

In our last theorem we describe an update rule that assigns to each new information the least tolerant consequence.

Theorem 8.3 *Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure that satisfies (5), and let $\mathcal{G} \mapsto \rho^\star_{\mathcal{G}} \in \mathcal{R}^{\rightarrow}_{\mathcal{G}}$ be the update rule with*

$$\rho^\star_{\mathcal{G}}(X) := \rho_{\mathcal{A}^\star_{\mathcal{G}}}(X) := \operatorname{ess\,inf}\{X' \in \mathcal{X}_{\mathcal{G}} \mid X + X' \in \mathcal{A}^\star_{\mathcal{G}}\} \quad \forall\ X \in \mathcal{X},$$

where the acceptance set of $\rho^\star_{\mathcal{G}}$ is given by

$$\mathcal{A}^\star_{\mathcal{G}} = \bigcap_{\rho_{\mathcal{G}} \in \mathcal{R}^{\rightarrow}_{\mathcal{G}}} \mathcal{A}_{\mathcal{G}}.$$

Every conditional convex risk measure $\rho^\star_{\mathcal{G}}$ is a consequence of ρ and $\mathcal{G} \subset \mathcal{F}$ and least tolerant in $\mathcal{R}^{\rightarrow}_{\mathcal{G}}$. Moreover, $\rho^\star_{\mathcal{G}}$ is also induced by the conditional convex hull of $\mathcal{A}(\mathcal{G})$

$$\operatorname{conv}_{\mathcal{G}}\mathcal{A}(\mathcal{G}) := \{\lambda X + (1-\lambda)Y \mid X, Y \in \mathcal{A}(\mathcal{G}),\ \lambda \in \mathcal{X}_{\mathcal{G}},\ 0 \leq \lambda \leq 1\}. \tag{20}$$

Proof: The intersection of acceptance sets is an acceptance set which generates a conditional convex risk measure. Hence, the update rule is well defined. It always assigns the least tolerant consequence of ρ and $\mathcal{G} \subset \mathcal{F}$.

Now, we consider the conditional convex hull of $\mathcal{A}(\mathcal{G})$. This set inherits the properties of $\mathcal{A}(\mathcal{G})$, cf. remark 4.1. It is solid and conditionally convex. By sensitivity of ρ, we have $\operatorname{ess\,inf}\ \mathcal{A}(\mathcal{G}) \cap \mathcal{X}_{\mathcal{G}} = 0$ which yields, $\operatorname{ess\,inf}\ \operatorname{conv}_{\mathcal{G}}\mathcal{A}(\mathcal{G}) \cap \mathcal{X}_{\mathcal{G}} = 0$. By proposition 2.5 in [5], $\operatorname{conv}_{\mathcal{G}}(\mathcal{A}(\mathcal{G}))$ generates a conditional convex risk measure $\rho'_{\mathcal{G}}$. Clearly, $\rho'_{\mathcal{G}} \in \mathcal{R}^{\rightarrow}_{\mathcal{G}}$ because $\mathcal{A}(\mathcal{G}) \subset \operatorname{conv}_{\mathcal{G}}\mathcal{A}(\mathcal{G}) \subset \mathcal{A}'_{\mathcal{G}}$. Since every acceptance set is conditionally convex, it holds $\operatorname{conv}_{\mathcal{G}}\mathcal{A}(\mathcal{G}) \subset \mathcal{A}'_{\mathcal{G}} \subset \mathcal{A}^\star_{\mathcal{G}}$. Hence, $\rho'_{\mathcal{G}} \preceq \rho^\star_{\mathcal{G}}$. By minimality of $\rho^\star_{\mathcal{G}}$ within the class of all consequences, both conditional convex risk measures are equal. □

9 Conclusions

We investigated different degrees of time consistency arising in the backward approach to dynamic risk assessment. We then introduced an alternative forward condition of consecutivity. We compared these approaches and illustrated them by the class of robust shortfall risk measures.

The main purpose of the paper was to discuss the problem of how to update a convex risk measure when new information arrives and to come up with a reasonable update rule. Only a few risk measures can be updated in a strongly time consistent sense. Therefore, we suggested to weaken the update criterion. We showed that a weakly consistent updating is possible in several ways, and we characterised the most tolerant weakly consistent update of a convex risk measure.

Finally, we proved that updating can also well be done under the forward criterion of consecutivity. Many consecutive update rules exist. In this class, we proposed the least tolerant risk measure as a reasonable update.

References

[1] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. Coherent measures of risk. *Mathematical Finance 9*, 3 (1999), 203–228.

[2] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., and Ku, H. Coherent multiperiod risk adjusted values and Bellman's principle. *Working Paper* (2003).

[3] Cheridito, P., Delbaen, F., and Kupper, M. Dynamic Monetary Risk Measures for Bounded Discrete-Time Processes. *Electronic Journal of Probability 11*, 3 (2006), 57–106.

[4] Dempster, A. P. Upper and Lower Probabilities Induced by a Multivalued Mapping. *The Annals of Mathematical Statistics 38*, 2 (1967), 325–339.

[5] Detlefsen, K., and Scandolo, G. Conditional and dynamic convex risk measures. *Finance and Stochastics 9*, 4 (2005), 539–561.

[6] Epstein, L. G., and Le Breton, M. Dynamically Consistent Beliefs Must Be Bayesian. *Journal of Economic Theory 61* (1993), 1–22.

[7] Fagin, R., and Halpern, J. Y. A new approach to updating beliefs. *Uncertainty in Artificial Intelligence 6* (1991), 347–374.

[8] Föllmer, H., and Penner, I. Convex risk measures and the dynamics of their penalty functions. *to appear in Statistics & Decisions* (2005).

[9] Föllmer, H., and Schied, A. Convex measures of risk and trading constraints. *Finance and Stochastics 6*, 4 (2002), 429–447.

[10] Föllmer, H., and Schied, A. Robust preferences and convex measures of risk. In *Advances in Finance and Stochastic. Essays in Honour of Dieter Sondermann.* Springer-Verlag, Berlin, 2002, pp. 39–56.

[11] Föllmer, H., and Schied, A. *Stochastic Finance: An Introduction in Discrete Time, 2^{nd} Edition.* Walter de Gruyter, Berlin - New York, 2004.

[12] Frittelli, M., and Rosazza Gianin, E. Putting Order in Risk Measures. *Journal of Banking and Finance 26*, 7 (2002), 1473–1486.

[13] Gilboa, I., and Schmeidler, D. Updating Ambiguous Beliefs. *Journal of Economic Theory 59*, 1 (1993), 33–49.

[14] Klöppel, S., and Schweizer, M. Dynamic utility indifference valuation via convex risk measures. *Working Paper, ETH Zürich* (2005).

[15] Riedel, F. Dynamic Coherent Risk Measures. *Stochastic Processes and Their Applications 112* (2004), 185–200.

[16] Roorda, B., and Schumacher, H. Time Consistency Conditions for Acceptability Measures, with an Application to Tail Value at Risk. *Working Paper* (2006).

[17] Schied, A. Optimal investments for risk- and ambiguity-averse preferences: a duality approach. *SFB 649 Discussion Paper* (2005).

[18] Shafer, G. *A Mathematical Theory of Evidence.* Princeton University Press, 1976.

[19] Tutsch, S. Konsistente und konsequente dynamische Risikomaße und das Problem ihrer Aktualisierung. *Dissertation, Humboldt-Universität zu Berlin* (2006).

[20] Walley, P. Coherent Lower (And Upper) Probabilities. *Statistics Research Report, University of Warwick, Coventry, U.K. 22* (1981).

[21] Weber, S. Distribution-Invariant Risk Measures, Information, and Dynamic Consistency. *Mathematical Finance 16*, 2 (2006), 419–441.

[22] Yoo, K.-R. The iterative law of expectation and non-additive probability measures. *Economics Letters 37* (1991), 145–149.