

Investment strategies in the long run with proportional transaction costs and HARA utility function

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INVESTMENT STRATEGIES IN THE LONG RUN WITH PROPORTIONAL TRANSACTION COSTS AND HARA UTILITY FUNCTION

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ABSTRACT. We consider an agent who invests in a stock and a money market in order to maximize the asymptotic behaviour of expected utility of the portfolio market price in the presence of proportional transaction costs. The assumption that the portfolio market price is a geometric Brownian motion and the restriction to utility function with hyperbolic absolute risk aversion (HARA) enable us to evaluate interval investment strategies. It is shown that the optimal interval strategy is also optimal among a wide family of strategies and that it is optimal also in a time changed model in case of logarithmic utility.

KEY WORDS: Trading strategies, transaction costs, asymptotic utility

Mathematics Subject Classification (1991): 60H30, 60G44, 91B28

1. INTRODUCTION

One of possible approaches to the problem of investment is to maximize the expected value of certain transformation of investor's wealth at a certain time in the future. It is reasonable to assume that such a transformation should be strictly increasing and concave and it is referred as a utility function. One of the most desirable such a function is the logarithmic one and it dates at least to Daniel Bernoulli in the eighteen century. It is known as Kelly criterion, see Kelly (1956), whose objective was to maximize the exponential growth rate rather than to use any utility function. Breiman (1961), Algoet and Cover (1988) showed that maximizing logarithmic utility leads to asymptotically maximal growth rate and asymptotically minimal expected time

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to reach a presigned goal. Bell and Cover (1988) showed that the expected log-optimal portfolio is also game theoretically optimal in a single play or in multiple plays of the stock market for a wide variety of pay off functions. Browne and Whitt (1996) used Bayesian approach in order to derive optimal gambling and investment policies for cases in which the underlying stochastic process has parameter values that are unobserved random variables. For further properties of Kelly criterion see Bell and Cover (1980), Rotando and Thorp (1992), Thorp (1997), Janeček (1999). Although this criterion has a lot of desirable properties, Samuelson (1971) and Thorp (1975) showed that maximizing geometric mean does not mean to end with a higher utility after a long time of investment. It is sufficient to consider other than logarithmic utility function with hyperbolic absolute risk aversion (HARA).

This paper is devoted to the simplest problem of investment in the presence of proportional transaction costs. We assume that the stock market price behaves as a geometric Brownian motion and we consider an investor who does not consume, but he/she withdraw from the market at the end of a very large time horizon. The classical approach to the problem of investment is to maximize the expected “present value” of future consumption over a finite or infinite horizon. This is the approach chosen by Merton (1971) in case of zero transaction costs and it leads to the strategy that keeps a constant proportion of total wealth held in the stock. This proportion is called Merton proportion and it is denoted by θ_γ here. For more general problems see Karatzas, Lehoczky, Sethi and Shreve (1986), Karatzas (1989). Magill and Constantinides (1976) formulated the problem in the presence of transaction costs and conjectured that the proportion of the total wealth invested in the stock should be kept within a certain interval. This problem was solved under restrictive conditions by Davis and Norman (1990) and analyzed by Shreve and Soner (1994). Constantinides (1986) numerically computed the effect of transaction costs on the value function for the problem and the width of the no-transaction region. His conjecture has been made precise by formal power series expansions in a variety models. A rigorous justification for the leading term in the expansion is given in Janeček and Shreve (2004). Morton and Pliska (1995) studied optimal portfolio management policies for an investor who must pay a transaction cost equal to a fixed fraction of his portfolio value each time he trades. Atkinson and Wilmott (1995) analyzed this model for the case of small transaction costs. The model with fixed and proportional transaction costs and multiple risky assets is studied in Liu (2004) from the point of view of a constant absolute risk aversion (CARA) investor.

Another approach is to consider a model without consumption and to maximize the expected utility at a certain time in the future. This problem leads to a variational inequality studied by Zhu (1991), Zhu

(1992) and solved numerically by Akian, Sulem and Séquier (1995) and Akian, Sulem, Séquier and A. Aboulalaa (1995).

Finally, the third approach is to maximize the asymptotic behaviour of expected utility as the end of the time horizon goes to infinity. This approach was chosen by Akian, Sulem and Taksar (2001) in case of logarithmic utility function. They also showed that such a problem can be interpreted as a limiting case of the classical investment-consumption problem and provide an explicit solution of one-risky-asset problem in case of logarithmic utility function in subsection 9.2.

Our approach is the third one and it is based on our ability to evaluate interval strategies in case of HARA utility functions. We find the optimal one and show that the restriction to certain strategies has not excluded the best one. The corresponding optimization problem is maximization of the growth rate of certainty equivalent CE of investor's wealth, see (2.3) for its specification.

2. NOTATION AND MODEL SET-UP

Suppose that the stock market price X_t is a geometric Brownian motion, driven by a Brownian motion $(W_t, t \geq 0)$, with

$$(2.1) \quad dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0 > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Denote by Y_t the portfolio market price at time $t \geq 0$ and by G_t the position of our investor in the market, i.e. G_t is the proportion of total wealth held in the stock at time $t \geq 0$. The investor may invest in a risky and non-risky asset. The amount of money invested in non-risky asset is equal to $Y_t(1 - G_t)$. Further, denoting H_t the number of shares present in the portfolio, we can express the stock part of the portfolio market price in two following forms $G_t Y_t = H_t X_t$.

We assume that the deposit part is not discounted. We suppose that we pay $(1 + b)$ -multiple of the stock market price in order to obtain the stock. On the other hand, we obtain $(1 - c)$ -multiple of the stock market price, when we sell it. We consider $b \in (0, \infty)$ and $c \in (0, 1)$. When we buy or sell the stock, the following value remains the same,

$$Y_t(1 + bG_t) = Y_t + bH_t X_t, \quad Y_t(1 - cG_t) = Y_t - cH_t X_t,$$

respectively. In particular, the investor is able to withdraw from the market with positive portfolio market price after withdrawing if and only if the position $G_t \in (-1/b, 1/c)$. We require a little bit more, we restrict ourselves to such strategies that G_t does not leave a compact set in $(-1/b, 1/c)$ and that $EY_t^\delta < \infty$ holds for every $\delta < 0$ and $t \geq 0$. This can be viewed as technical assumption and these conditions determine the class of admissible strategies. The controls are the processes describing the amount of shares bought and sold on $[0, t)$ denoted by

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$H_1^+(t)$ and $H_1^-(t)$, respectively. Then the portfolio market price can be computed by (3.2).

We will consider only utility functions with hyperbolic absolute risk aversion (HARA) $\mathcal{U}_\gamma(x) = \frac{1}{\gamma}x^\gamma$ if $\gamma < 0$ and $\mathcal{U}_0(x) = \ln x$. The case $\gamma \in (0, 1]$ is omitted, since it leads to a utility function bounded from below. Further denote $e_\gamma(x) := \mathcal{U}_\gamma(e^x)$, then $\mathcal{U}_\gamma(y) = e_\gamma(\ln y)$. For simplicity, we assume that $Y_0 = y_0 > 0$ is a deterministic random variable.

The aim of this paper is to find $f \in C^2(-1/b, 1/c)$, $\nu \in \mathbb{R}$ and a special strategy such that (2.2) is a supermartingale when considering any admissible strategy and such that

$$(2.2) \quad e_\gamma(\ln Y_t - f(G_t) - \nu t) \quad \text{is a martingale,}$$

if we consider the special one. It follows from the moreover part of theorem 5.7 that the special strategy is admissible. Our approach is associated with the following optimization problem

$$(2.3) \quad \max \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{CE}_t, \quad \text{where} \quad \mathbf{CE}_t := \mathcal{U}_\gamma^{-1} E \mathcal{U}_\gamma(Y_t)$$

over all admissible strategies, where \liminf can be replaced also by \limsup . The approach (2.2) enable us to prove a little bit more, than that the special strategy solves (2.3), see (2.5), (2.4) and (2.6).

Remark 2.1 Let us consider two admissible strategies with the portfolio market prices Y_t and \hat{Y}_t such that the second one is special in sense (2.2). Since we assume that f is a continuous function on $(-1/b, 1/c)$ and that G_t does not leave a compact set in $(-1/b, 1/c)$, we get that $f(G_t)$ is a bounded process, which balances the infinitesimal increment of expected utility corresponding to different current positions G_t . In what follows, we show that ν is the desired maximum (2.3) reached by the special strategy. Moreover, we will see that

$$(2.4) \quad E \mathcal{U}_\gamma(Y_t) \leq \mathcal{U}_\gamma(\exp\{\nu t + O(1)\}) = e_\gamma(\nu t + O(1))$$

$$(2.5) \quad E \mathcal{U}_\gamma(\hat{Y}_t) = \mathcal{U}_\gamma(\exp\{\nu t + O(1)\}) = e_\gamma(\nu t + O(1))$$

as $t \rightarrow \infty$, i.e. the certainty equivalent of the portfolio market price \hat{Y}_t corresponding to the special strategy is $\exp\{\nu t + O(1)\}$ as $t \rightarrow \infty$ and ν is the rate of this exponential trend.

The certainty equivalent corresponding to a random variable V and a utility function \mathcal{U} is a value v giving the same expected utility as V , i.e. $\mathcal{U}(v) = E \mathcal{U}(V)$.

Let \hat{G}_t stand for the position corresponding to the special strategy. If $\gamma = 0$ or if $\gamma < 0$, then the martingale property (2.2) with $\hat{U}_t :=$

$\ln \hat{Y}_t - f(\hat{G}_t) - \nu t$ gives that

$$\begin{aligned} E \ln \hat{Y}_t &= \nu t + E[f(\hat{G}_t) - f(\hat{G}_0)] + \ln y_0 = \nu t + O(1), \\ EU_\gamma(\hat{Y}_t) &= E e_\gamma(\ln \hat{Y}_t - f(\hat{G}_t)) e^{O(1)} = \exp\{\gamma[\nu t + O(1)]\} E e_\gamma(\hat{U}_t) \\ &= \exp\{\gamma[\nu t + O(1)]\} E e_\gamma(\hat{U}_0) = e_\gamma(\nu t + O(1)) \end{aligned}$$

respectively, and it is nothing else but (2.5). Similarly, the supermartingale property gives (2.4). In particular, we have that

$$(2.6) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{U}_\gamma^{-1} E \mathcal{U}_\gamma(Y_t) \leq \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{U}_\gamma^{-1} E \mathcal{U}_\gamma(\hat{Y}_t) = \nu.$$

See corollary 6.5 and theorem 4.4 that there are $\alpha < \beta$ such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and $f \in C^2(-1/b, 1/c), \nu \in \mathbb{R}$ such that $e_\gamma(\ln Y_t - f(G_t) - \nu t)$ is a supermartingale when considering any admissible strategy and that it is a martingale in case that we just keep the position G_t within the interval $[\alpha, \beta]$, i.e. we do not trade when $G_t \in (\alpha, \beta)$, but we trade in order to ensure that $G_t \in [\alpha, \beta]$ holds for every $t > 0$.

The values α, β can be obtained from (6.7), the corresponding functions ξ_\pm are defined in remark 4.2 and the parameter ω_γ is the unique root of the equation $\mathcal{I}(\omega_\gamma) + \ln \frac{1+b}{1-c} = 0$ on certain interval $(0, \tilde{\omega}_\gamma)$, where \mathcal{I} is defined by (6.4). The rate ν of the exponential trend of the certainty equivalent is given by the formula $\nu = \frac{1-\gamma}{2} \sigma^2(\theta_\gamma^2 - \omega_\gamma^2)$.

The following section is devoted to the dynamics in a model, where the agent can invest into one or more risky assets, but pays the transaction costs only for the first one, which of course covers the case of this paper and also of the forthcoming one.

Section 4 is devoted to the optimality conditions in terms of the balancing function f and it culminates by theorem 4.4. In section 5, we evaluate interval strategies and we prove the main theorem of this section, theorem 5.7. In section 6, we prove existence of certain value of parameter ω_γ , which determine by (6.7) the optimal strategy. The last section contains concluding remarks.

3. DYNAMICS IN MULTIDIMENSIONAL MODEL

We are going to consider the case when we have more than one stock in the market and we pay the transaction costs only for the first one in order to ensure that the statements are so general that they can be used also in Dostál (2006). We write \mathbf{x}^\top for the transposition of \mathbf{x} and f' for the derivative of f . We refer the reader, who is not familiar with stochastic integration, to Chapter 3 in Karatzas and Shreve (1991) for the corresponding theory.

We switch to the n -dimensional model of stock market price given by (3.1). Let \mathcal{F}_t be an augmented filtration and $W(t)$ be an n -dimensional

\mathcal{F}_t -Wiener process. Further, we assume that the stock market price is an n -dimensional \mathcal{F}_t -semimartingale with stochastic differential

$$(3.1) \quad dX(t) = \mathbb{X}(t)\mu dt + \mathbb{X}(t)\Sigma^{\frac{1}{2}} dW(t), \quad X(0) = \mathbf{x} \in (0, \infty)^n,$$

where $\mathbb{X}(t) := \text{diag } X(t)$ and $\Sigma^{1/2} \in \mathbb{R}^{n \times n}$ is a positively definite matrix such that $\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}} =: \Sigma$ and $\mu \in \mathbb{R}^n$. Further, denote by $H(t) = (H_1(t), \dots, H_n(t))^{\top}$ the vector of numbers of shares of each stock and by $G(t) = (G_1(t), \dots, G_n(t))^{\top}$ the vector of the positions of our investor in the market. Put $\mathbb{G}(t) := \text{diag } G(t)$, $\mathbb{H}(t) := \text{diag } H(t)$. Then $Y(t)G(t) = \mathbb{H}(t)X(t) = \mathbb{X}(t)H(t)$.

We restrict ourselves to the strategies such that $Y(t) > 0$ and $G_1(t) \in (-1/b, 1/c)$ hold for every $t \geq 0$ almost surely and we always assume that the transaction costs at time t are paid at the next moment after t . Let $H_1^+(t)$ and $H_1^-(t)$ denote the sum of shares of the first stock bought and sold on the time interval $[0, t)$, respectively. These processes will be referred to as the control processes. We assume that these processes are non-decreasing \mathcal{F}_t -adapted left-continuous with right-hand limits. Further, we assume that $H_i(t)$, $G_i(t)$ and $Y(t)$ are locally bounded \mathcal{F}_t -progressive measurable processes for $1 \leq i \leq n$. We are going to show some basic facts such as $H_1^+(t)$, $H_1^-(t)$ are finite almost surely.

Lemma 3.1 *Let $Y(t) > 0$, $G_1(t) \in (-1/b, 1/c)$ hold for every $t \geq 0$ almost surely. Then $H_1^+(t)$, $H_1^-(t) < \infty$ and*

$$(3.2) \quad Y(t) = y_0 + \int_0^t H(s)^{\top} dX(s) - bX_1(s) dH_1^+(s) - cX_1(s) dH_1^-(s)$$

hold for every $t \geq 0$ almost surely, provided that $L := \int \mathbb{X}(t)^{-1} dX(t)$ is a continuous \mathcal{F}_t -semimartingale. We do not assume (3.1) now.

Proof. Obviously, $\tilde{Y} := y_0 + \int H(t)^{\top} dX(t) \geq Y$ hold at each $t \geq 0$ almost surely and $\tilde{Y}(t)$ is the portfolio market price at time t corresponding to zero transaction tax. Further, $T(t) := \int_0^t X_1(s)[b dH_1^+(s) + c dH_1^-(s)]$ denotes the total transaction costs on $[0, t)$. By definition of the portfolio market price $Y(t) = \tilde{Y}(t) - T(t)$ and therefore (3.2) holds for every $t \geq 0$ almost surely, since we pay the transaction costs corresponding to time t at the next moment after t . Since $\tilde{Y}(t) < \infty$ and $Y(t) > 0$ hold for every $t \geq 0$, we obtain that $T(t) = \tilde{Y}(t) - Y(t) < \infty$ holds for every $t \geq 0$ almost surely and therefore $H_1^+(t) \leq T(t)/[b \min_{s \in [0, t]} X_1(s)] < \infty$ and $H_1^-(t) \leq T(t)/[c \min_{s \in [0, t]} X_1(s)] < \infty$ hold for every $t \geq 0$ almost surely. \square

Remark 3.2 We are going to compute with stochastic differentials as every integrator is a continuous process. It happens in case that the control processes $H_1(t)^+$ and $H_1(t)^-$ have no jumps. If this is not the case, we denote $\Delta^{\pm} := \{s \in [0, \infty), H_1^{\pm}(s) \neq H_1^{\pm}(s_+)\}$ the set of all points, where $H_1^+(t)$ and $H_1^-(t)$ jumps, respectively. We

restrict ourselves to the strategies that do not sell and buy the first stock at the same time. In particular, we assume that Δ^+ and Δ^- are disjoint sets. If we buy or sell the stock, then $Y(t)(1 + bG_1(t))$ or $Y(t)(1 - cG_1(t))$ remains the same before and after the transaction, respectively. Further, we put

$$(3.3) \quad X(s, u) := X(s), \quad G(s, u) := (1 - u)G(s) + uG(s_+)$$

$$(3.4) \quad H_i(s, u) := \frac{Y(s, u)G_i(s, u)}{X_i(s, u)}, \quad Y(s, u) := Y(s) \frac{\vartheta_{\pm}(G_1(s, u))}{\vartheta_{\pm}(G_1(s))}$$

if $s \in \Delta^{\pm}$ and $u \in [0, 1]$, where $\vartheta_+(x) := b/(1 + bx)$ and $\vartheta_-(x) := c/(1 - cx)$. We have chosen formula (3.4) to define $Y(s, u)$ in order to ensure that the values $Y(s, u)/\vartheta_{\pm}(G_1(s, u))$ remain constant in u if $s \in \Delta^{\pm}$ in order to be able to interpret $Y(s, u)$ as the portfolio market price at time s provided that we have executed only those transactions at time s that change the first position G_1 from $G_1(s, 0)$ to $G_1(s, u)$ and provided that we pay the transaction costs immediately.

If M is one of the processes X, Y, G_1, H_1 or their function and $f : \mathbb{R}^{3n+2} \rightarrow \mathbb{R}$ is a Borel measurable function, we define the following integral $\int_s^t f(v, X(v), Y(v), G(v), H(v)) dM(v)$ as

$$(3.5) \quad \int_s^t f(\mathcal{X}(v)) dM^c(v) + \sum_{v \in [s, t) \cap \Delta^{\pm}} \int_0^1 f(\mathcal{X}(v, u)) M(v, du),$$

whenever (3.5) is defined, where M^c denotes the continuous part of M , $\mathcal{X}(v, u) := (v, X(v, u), Y(v, u), G(v, u), H(v, u))$ and where $\mathcal{X}(v)$ stands for $\mathcal{X}(v, 0)$. Further, we define

$$G_1^+(t) := \int_0^t Y(s)^{-1}(1 + bG_1(s))X_1(s) dH_1^+(s)$$

$$G_1^-(t) := \int_0^t Y(s)^{-1}(1 - cG_1(s))X_1(s) dH_1^-(s)$$

and we extend the definition (3.5) also to the case when $M = G_1^+$ or $M = G_1^-$. To justify using Itô formula for continuous semimartingales, we are to show that

$$\Delta F(\mathcal{X}(v)) := F(\mathcal{X}(v_+)) - F(\mathcal{X}(v)) = \int_0^1 \nabla F(\mathcal{X}(v, u)) \mathcal{X}(v, du)^{\top}$$

if $v \in \Delta^{\pm}$ and $F \in C^1(\mathcal{X}(v, u), u \in [0, 1])$, but it follows immediately since the components of $\mathcal{X}(v, u)$ are continuous processes on $[0, 1]$ in u with finite variation. Note that this definition of integration is consistent with the usual definition if the integrand is the stock market price or its function, since it does not change during transactions, and therefore we have not changed the meaning of the statement and of the proof of lemma 3.1. Further, we will abbreviate the notation

$$h_{\pm}(G_1(t)) * dG_1^{\pm}(t) := h_+(G_1(t)) dG_1^+(t) + h_-(G_1(t)) dG_1^-(t)$$

whenever h_+ and h_- are continuous functions on $(-1/b, 1/c)$. Then we obtain from (3.2) and the definition of G_1^\pm that

$$(3.6) \quad Y(t)^{-1}dY(t) = G(t)^\top \mathbb{X}(t)^{-1}dX(t) - \vartheta_\pm(G_1(t)) * dG_1^\pm(t).$$

Further, denote $e_1 \in \mathbb{R}$ the column vector consisting of 1 in the first row and 0 in the remaining ones.

Lemma 3.3 *Let $Y(t) > 0$, $G_1(t) \in (-1/b, 1/c)$, let $L(t)$ be a continuous \mathcal{F}_t -semimartingale such that $\mathbb{X}(t)^{-1}dX(t) = dL(t)$. Then*

$$dG_1(t) = e_1^\top [\mathbb{G}(t) - G(t)G(t)^\top] [dL(t) - \sum_{j=1}^n G_j(t) d\langle L, L_j \rangle] \pm dG_1^\pm(t).$$

In particular, if (3.1) holds, we have that

$$(3.7) \quad dG_1(t) = B_1(G(t)) dt + S_1(G(t)) dW(t) + dG_1^+(t) - dG_1^-(t),$$

where $B_1(x) := e_1^\top B(x)$, $S_1(x) := e_1^\top S(x)$ and

$$B(x) = [\mathbb{x} - \mathbb{x}\mathbb{x}^\top][\mu - \Sigma\mathbb{x}], \quad S(x) = [\mathbb{x} - \mathbb{x}\mathbb{x}^\top]\Sigma^{\frac{1}{2}}.$$

Proof. By Itô formula, we obtain from (3.6) that

$$(3.8) \quad Y(t) dY(t)^{-1} = -Y(t)^{-1}dY(t) + Y(t)^{-2}d\langle Y \rangle(t)$$

$$(3.9) \quad = -G(t)^\top [dL(t) - \sum_{j=1}^n G_j(t) d\langle L, L_j \rangle(t)] + \vartheta_\pm(G_1(t)) * dG_1^\pm(t).$$

Further, we obtain from the definition of $G_1^\pm(t)$ that

$$(3.10) \quad X_1(t) dH_1(t) = \pm Y(t)[1 \mp G_1(t)\vartheta_\pm(G_1(t))] * dG_1^\pm(t)$$

and therefore $G_1(t)Y(t) dY(t)^{-1} + Y(t)^{-1}X_1(t) dH_1(t)$ is equal to

$$(3.11) \quad -G_1(t)G(t)^\top [dL - \sum_{j=1}^n G_j(t) d\langle L, L_j \rangle] + dG_1^+(t) - dG_1^-(t).$$

Further, $Y(t)^{-1}H_1(t) dX_1(t) = e_1^\top \mathbb{G}(t) dL(t)$ and therefore

$$(3.12) \quad H_1(t) d\langle X_1, Y^{-1} \rangle(t) = -e_1^\top \mathbb{G}(t) \sum_{j=1}^n G_j(t) d\langle L, L_j \rangle(t).$$

Since $G_1(t) = Y(t)^{-1}H_1(t)X_1(t)$ and $H_1(t)$ is of locally finite variation, we obtain by Itô formula that

$$(3.13) \quad dG_1(t) = Y(t)^{-1}X_1(t) dH_1(t) + Y(t)^{-1}H_1(t) dX_1(t)$$

$$(3.14) \quad + G_1(t)Y(t) dY(t)^{-1} + H_1(t) d\langle X_1, Y^{-1} \rangle(t)$$

$$(3.15) \quad = e_1^\top [\mathbb{G}(t) - G(t)G(t)^\top] [dL(t) - \sum_{j=1}^n G_j(t) d\langle L, L_j \rangle] \pm dG_1^\pm(t).$$

□

Lemma 3.4 *Let $f \in C^2(-1/b, 1/c)$ and $\nu \in \mathbb{R}$. Let $Y(t) > 0, G_1(t) \in (-1/b, 1/c)$ hold for every $t \geq 0$ almost surely. Then $U_t := \ln Y_t - f(G_t) - \nu t$ is an \mathcal{F}_t -semimartingale with*

$$(3.16) \quad \frac{de_\gamma(U(t))}{\exp\{\gamma U_t\}} = d_f^\nu(G(t)) dt + v_f(G(t)) dW(t) + \delta_\pm^f(G_1(t)) * dG_1^\pm(t),$$

where $v_f(x) := x^\top \Sigma^{1/2} - f'(x) S_1(x), x := e_1^\top x, \delta_\pm^f(x) := -\vartheta_\pm(x) \mp f'(x)$

$$d_f^\nu(x) := d(x) - \nu - f'(x) e_1^\top \tilde{B}(x) - \frac{1}{2} [f''(x) - \gamma f'(x)^2] S_1(x) S_1(x)^\top$$

$$\tilde{B}(x) := [x - x x^\top] [\mu - (1 - \gamma) \Sigma x], \quad d(x) := \mu^\top x - \frac{1 - \gamma}{2} x^\top \Sigma x.$$

Proof. By Itô formula, $e^{-\gamma U(t)} de_\gamma(U(t)) = dU(t) + \frac{\gamma}{2} d\langle U \rangle(t)$ equals to

$$\frac{de_\gamma(\ln Y(t))}{Y(t)^\gamma} - df(G_1(t)) - \nu dt + \frac{\gamma}{2} d\langle f(G_1) \rangle(t) - \gamma d\langle \ln Y, f(G_1) \rangle(t).$$

By Itô formula, we easily obtain from (3.6) that

$$Y(t)^{-\gamma} de_\gamma(\ln Y(t)) = d(G(t)) dt + G(t)^\top \Sigma^{1/2} dW(t) - \vartheta_\pm(G_1(t)) * dG_1^\pm(t)$$

and therefore it is seen from (3.7) that the terms containing $dW(t)$ agree. Similarly, we can see that the terms containing $dG_1^\pm(t)$ also agree. Now, it is sufficient to compute

$$\begin{aligned} \gamma \langle \ln Y, f(G_1) \rangle(t) &= \gamma f'(G_1(t)) e_1^\top S(G(t)) \Sigma^{\frac{1}{2}} G(t) dt \\ &= f'(G_1(t)) e_1^\top [\tilde{B}(G(t)) - B(G(t))] dt \end{aligned}$$

in order to verify that the terms containing dt also agree. \square

Lemma 3.5 *Let $\mathcal{K} \subseteq (-1/b, 1/c) \times \mathbb{R}^{n-1}$ be a compact set. Let $Y(t) > 0$ and $G(t) \in \mathcal{K}$ hold for every $t \geq 0$ almost surely and $f \in C^2(-1/c, 1/b)$.*

(i) *Let $-1/b < \alpha < \beta < 1/c$ and $\nu \in \mathbb{R}$. Further assume that $G_1(t) \in [\alpha, \beta], d_f^\nu(G(t)) = 0$ and*

$$\int_0^t \delta_+^f(G_1(s)) dG_1^+(s) = \int_0^t \delta_-^f(G_1(s)) dG_1^-(s) = 0$$

hold for every $t \geq 0$ almost surely. Then $EY(t)^\delta < \infty$ holds for every $\delta < 0$ and $t \geq 0$. Further, $e_\gamma(U(t)) = V(t)$ holds for every $t \geq 0$ almost surely, where V is given by (3.17).

(ii) *Let $EY(t)^\delta < \infty$ hold for every $\delta < 0$ and $t \geq 0$. Then*

$$(3.17) \quad V := e_\gamma(U(0)) + \int \exp\{\gamma U(s)\} v_f(G(s)) dW(s)$$

is an \mathcal{F}_t -martingale.

Proof. (i) By lemma 3.4, we get that $de_\gamma(U(t)) = e^{\gamma U(t)} dZ(t)$, where $Z := U(0) + \int v_f(G(s)) dW(s)$ and therefore $e_\gamma(U(t)) = V(t)$ holds for every $t \geq 0$ almost surely. If $\gamma = 0$, we obtain that $U = e_\gamma(U) = V = Z$

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hold almost surely. If $\gamma < 0$, we get that $\exp\{\gamma U(t)\} = \exp\{\gamma Z(t) - \frac{\gamma^2}{2}\langle Z \rangle(t)\}$ holds for every $t \geq 0$ almost surely. Hence, we obtain that

$$(3.18) \quad \exp\{\delta U(t)\} = \exp\{\delta Z(t) - \delta^2 \langle Z \rangle(t)/2\} \exp\{\delta(\delta - \gamma)\langle Z \rangle(t)/2\}$$

holds for every $t \geq 0$ almost surely and (3.18) is a product of an exponential martingale and a process which is bounded on $[0, t]$ for every $t \geq 0$ almost surely, since $v_f(G(t)) v_f(G(t))^\top$ is a bounded process. It follows from definition of $U(t) = \ln Y(t) - f(G_1(t)) - \nu t$ that

$$EY(t)^\delta \leq Ee^{\delta U(t)} \exp\{\delta(\min_{[\alpha, \beta]} |f| + \nu t)\} < \infty.$$

(ii) Obviously, We obtain from assumption and the inequality $e_\delta(x) \leq x$ that $-\infty < Ee_\delta(\ln Y(t)) \leq E \ln Y(t)$ hold for every $\delta < 0$ and therefore $\ln Y(t)$ integrable from below. Since $2G(t)^\top \mu - G(t)^\top \Sigma G(t) \leq \mu^\top \Sigma^{-1} \mu$ holds for every $t \geq 0$, we obtain from Itô formula that

$$(3.19) \quad d \ln Y(t) - \tilde{\nu} dt \leq G(t)^\top \Sigma^{\frac{1}{2}} dW(t), \quad \text{where } \tilde{\nu} := \mu^\top \Sigma^{-1} \mu/2.$$

By assumption $G(t)^\top \Sigma^{\frac{1}{2}}$ is a bounded process almost surely and therefore $\int G(t)^\top \Sigma^{\frac{1}{2}} dW(t)$ is a martingale. Hence, we get by (3.19) that $\ln Y(t) - \tilde{\nu} t$ is a supermartingale and Jensen inequality gives that $e_\delta(\ln Y(t) - \tilde{\nu} t) = \frac{1}{\delta} Y(t)^\delta e^{-\delta \tilde{\nu} t}$ is also a supermartingale for every $\delta < 0$, since it is integrable by assumption. In particular, $EY(s)^{2\gamma} \leq EY(t)^{2\gamma} e^{\eta(t-s)}$ holds for every $s \in [0, t]$, where $\eta := -2\gamma \tilde{\nu} \geq 0$. Further,

$$E\langle V \rangle(t) = E \int_0^t e^{2\gamma U(s)} v_f(G(s)) v_f(G(s))^\top ds \leq t K_t \cdot EY(t)^{2\gamma} e^{\eta t} < \infty,$$

where K_t is an upper bound of the following process $v_f(G(s)) v_f(G(s))^\top \cdot \exp\{-2\gamma[f(G_1(s)) + \nu s]\}$ on $[0, t]$. Hence, V_t is an L_2 -martingale as $e_\gamma(U(0))$ is a bounded random variable. \square

Remark 3.6 We are going to rewrite the main formulas of this section that are used later on into the one-dimensional form. First, we put $G_t^\pm := G_1^\pm(t)$. Then (3.6) and (3.7) can be rewritten into the form

$$(3.20) \quad Y_t^{-1} dY_t = G_t[\mu dt + \sigma_t dW_t] - \vartheta_+(G_t) dG_t^+ - \vartheta_-(G_t) dG_t^-,$$

$$(3.21) \quad dG_t = G_t(1 - G_t)[(\mu - \sigma^2 G_t) dt + \sigma dW_t] + dG_t^+ - dG_t^-,$$

respectively. In lemma 3.4, we get $v_f(x) = \sigma x[1 - (1 - x)f'(x)]$ and

$$(3.22) \quad -d_f^\nu = \frac{1}{2} S^2(x)[f''(x) - \gamma f'(x)^2] + \tilde{B}(x)f'(x) - d(x) + \nu$$

with $d(x) = \mu x - \frac{1-\gamma}{2} \sigma^2 x^2$ and

$$(3.23) \quad S(x) = \sigma x(1 - x), \quad \tilde{B}(x) = x(1 - x)[\mu - (1 - \gamma)\sigma^2 x].$$

4. DYNAMICS OF INTERVAL STRATEGIES IN ONE-DIMENSIONAL MODEL

We focus on such strategies that do not trade, when the position G_t is inside the interval (α, β) and such that buy or sell the stock in order to keep the position G_t within the interval $[\alpha, \beta]$. These strategies are called interval strategies here and denoted as $[(\alpha, \beta)]$. Their existence is an indivisible part of the investment-consumption theory, since the optimal strategies are of this type in this theory in case of proportional transaction costs, see also theorem 6.4 in Dostál (2006). In this section, we use only one-dimensional processes and so we can use the lower index for the time argument.

Let us look at the case when the transaction costs are zero. Let us recall that $\theta_\gamma := \frac{\sigma^2 \mu}{1 - \gamma}$ denotes the Merton proportion. It can be easily verified that $e_\gamma(\ln Y_t - \frac{1-\gamma}{2} \sigma^2 \theta_\gamma^2)$ is a supermartingale when considering any strategy that keeps the position within a compact interval in \mathbb{R} and it is a martingale when applying the strategy $[(\theta_\gamma, \theta_\gamma)]$. The strategy $[(\theta_\gamma, \theta_\gamma)]$ is no-trading if $\theta_\gamma = 0$ or $\theta_\gamma = 1$. In these two cases, the above mentioned strategy is also optimal in the presence of transaction costs and this is the reason why we consider only strategies $[(\alpha, \beta)]$ such that $\alpha < \beta$ and $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and the case $\theta_\gamma \notin \{0, 1\}$.

Theorem 4.1 *Let $-1/b < \alpha < \beta < 1/c, \nu \in \mathbb{R}$ be such that there exists $f \in C^2[\alpha, \beta]$ such that $d_f^\nu(x) = 0$ holds for every $x \in [\alpha, \beta]$ and $\delta_+^f(\alpha) = \delta_-^f(\beta) = 0$, then $e_\gamma(U_t)$ is an \mathcal{F}_t -martingale and*

$$\nu(\alpha, \beta) := \lim_{t \rightarrow \infty} \frac{1}{t} e_\gamma^{-1} E e_\gamma(\ln Y_t) = \nu$$

when applying the strategy $[(\alpha, \beta)]$. In particular, if $\alpha \leq \pi_+ < \pi_- \leq \beta$ are such that $\delta_+^f(\pi_+) = \delta_-^f(\pi_-) = 0$, then $\nu(\pi_+, \pi_-) = \nu(\alpha, \beta)$.

Proof. Obviously, f can be extended to an element of $C^2(-1/b, 1/c)$. By lemma 3.5, the process $e_\gamma(U_t)$ is an \mathcal{F}_t -martingale and therefore $e_\gamma^{-1} E e_\gamma(U_t) = e_\gamma^{-1} E e_\gamma(U_0) = o(t)$ as $t \rightarrow \infty$. Hence,

$$\lim_{t \rightarrow \infty} \frac{1}{t} e_\gamma^{-1} E e_\gamma(\ln Y_t) = \lim_{t \rightarrow \infty} \frac{1}{t} e_\gamma^{-1} E e_\gamma(\ln Y_t - f(G_t) - \nu t) + \nu = \nu.$$

□

Remark 4.2 Further, we introduce

$$\xi_+(x) := x \frac{1+b}{1+bx}, \quad \xi_-(x) := x \frac{1-c}{1-cx}.$$

If $x > 0$, then $\xi_+(x)$ denotes the proportion between total costs and the initial capital necessary for reaching the position $x > 0$ and $\xi_-(x)$ denotes the proportion between the income and final wealth of the investor when he/she withdraws from the market provided that his/her initial position was $x > 0$.

Lemma 4.3 *Let $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ be such that $\xi_+(\alpha) < \theta_\gamma < \xi_-(\beta)$ and such that $\nu(u, \beta) \neq \nu(\alpha, \beta) \neq \nu(\alpha, u)$ whenever $u \in (\alpha, \beta)$. Let $f \in C^2(-1/b, 1/c)$ and $\nu \in \mathbb{R}$ be such that*

$$(4.1) \quad \delta_+^f(y) = \delta_-^f(z) = d_f^\nu(x) = 0$$

hold for every $y \in (-1/b, \alpha]$, $z \in [\beta, 1/c)$ and $x \in [\alpha, \beta]$. Then

$$(4.2) \quad \delta_+^f(y) < 0, \quad \delta_-^f(z) < 0, \quad d_f^\nu(x) < 0$$

hold for every $y \in (\alpha, 1/c)$, $z \in (-1/b, \beta)$ and $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$.

Proof. Let $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$. It follows from the assumptions that $f''(x) = f'(x)^2$. Then

$$(4.3) \quad -d_f^\nu(x) = \frac{1-\gamma}{2} \sigma^2 [\theta_\gamma - x(1 - (1-x)f'(x))]^2 + \nu - \frac{1-\gamma}{2} \sigma^2 \theta_\gamma^2.$$

If $y \in (-1/b, \alpha)$ and $z \in (\beta, 1/c)$, then

$$\xi(y, f'(y)) = \xi_+(y) < \xi_+(\alpha) < \theta_\gamma < \xi_-(\beta) < \xi_-(z) = \xi(z, f'(z)),$$

where $\xi(x, h) := x[1 - (1-x)h]$, and therefore $d_f^\nu(y) < d_f^\nu(\alpha) = 0 = d_f^\nu(\beta) > d_f^\nu(z)$, i.e. $d_f^\nu(x) < 0$ holds for every $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$.

Obviously, $x \in (-1/b, 1/c) \mapsto -\vartheta_+(x) - \vartheta_-(x)$ is a negative function and it is equal to δ_-^f on $(-1/b, \alpha]$ and to δ_+^f on $[\beta, 1/c)$. Hence, we are now only to show that δ_+^f, δ_-^f are negative also on (α, β) . As we know, $\delta_+^f(\beta) < 0$ and $\delta_-^f(\alpha) < 0$. Since δ_+^f, δ_-^f are continuous functions, it is sufficient to show that there exists no $x \in (\alpha, \beta)$ such that $\delta_+^f(x) = 0$ or $\delta_-^f(x) = 0$. If such an x exists, then $f'(x) = -\vartheta_+(x)$ or $f'(x) = \vartheta_-(x)$. Then $\nu(\alpha, \beta)$ is by theorem 4.1 equal to $\nu(x, \beta)$ or $\nu(\alpha, x)$, respectively, which is not possible by assumption. \square

Theorem 4.4 *Let $-1/b < \alpha < \beta < 1/c, \nu \in \mathbb{R}, f \in C^2(-1/b, 1/c)$ be such that (4.1) hold for every $y \in (-1/b, \alpha]$, $z \in [\beta, 1/c)$ and $x \in [\alpha, \beta]$ and (4.2) for every $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$, $y \in (\alpha, 1/c)$ and $z \in (-1/b, \beta)$. Let us consider a strategy that keeps G_t within a compact interval in $(-1/b, 1/c)$ such that $EY_t^\delta < \infty$ holds for every $\delta < 0, t \geq 0$. Then $e_\gamma(U_t)$ is an \mathcal{F}_t -supermartingale and it is an \mathcal{F}_t -martingale if the strategy $[(\alpha, \beta)]$ is applied. Moreover, if $\gamma = 0$, then*

$$(4.4) \quad \nu = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{Y}_t \geq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t$$

holds almost surely, where \hat{Y}_t denotes the portfolio market price at time t corresponding to strategy $[(\alpha, \beta)]$ here.

Proof. If $[(\alpha, \beta)]$ is applied, then $e_\gamma(U_t)$ is an \mathcal{F}_t -martingale by theorem 4.1. Let us consider a strategy that keeps G_t within a compact interval in $(-1/b, 1/c)$ and such that $EY_t^\delta < \infty$ holds for every $\delta < 0$

and $t \geq 0$. By lemma 3.5, $V = e_\gamma(U_0) + \int e^{\gamma U_s} v_f(G_s) dW_s$ is an \mathcal{F}_t -martingale. By lemma 3.4

$$(4.5) \quad e_\gamma(U_t) - V_t = \int_0^t e^{\gamma U_s} [d_f^\nu(G_s) ds + \delta_\pm^f(G_s) * dG_s^\pm]$$

holds for every $t \geq 0$ almost surely. By assumption, the right-hand side of (4.5) is a non-increasing process starting from 0. In particular, $e_\gamma(U_t)$ is an integrable from above for all $t \geq 0$ also in case $\gamma = 0$. If $\gamma < 0$, $e_\gamma(U_t)$ is by definition of e_γ bounded from above. If $\gamma < 0$, it follows from assumption that $EY(t)^\delta < \infty$ holds for every $\delta < 0 \leq t$. If $\gamma = 0$, we obtain that $EU_t \geq e_\delta(U_t) > -\infty$ by the same assumption and the following inequality $x \geq e_\delta(x)$ whenever $\delta < 0$. Now, we are going to prove the moreover part of the statement. Since $v_f(G_s)$ is a bounded process, we get by BDG-inequality, see Proposition 15.7 in Kallenberg (1997), that there exists $C_2 \in (0, \infty)$ such that

$$P(\max_{s \leq t} |V_s| \geq \varepsilon t) \leq \frac{E \max_{s \leq t} |V_s|^2}{\varepsilon^2 t^2} \leq \frac{C_2}{\varepsilon^2 t^2} \int_0^t v_f^2(G_s) ds \rightarrow 0$$

as $t \rightarrow \infty$ whenever $\varepsilon > 0$. Hence, $V_t/t \rightarrow 0$ as $t \rightarrow \infty$ almost surely. Then (4.4) follows from (4.5) and the definition of $U_t = \ln Y_t - f(G_t) - \nu t$, since the right-hand side of (4.5) is a non-decreasing process generally and it is zero when applying the strategy $[(\alpha, \beta)]$. \square

5. EVALUATION OF INTERVAL STRATEGIES

In this section, we assume that $\alpha < \beta$ are fixed and satisfy $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$. Further, we put $\rho := (1 - \gamma)\theta_\gamma - \frac{1}{2} = \sigma^{-2}\mu - \frac{1}{2}$. First, we introduce two lemmas whose verification is left to the reader.

Lemma 5.1 *Let $\gamma < 0$. In case that $\nu \neq -\frac{\sigma^2 \rho^2}{2\gamma}$, the equation (5.1)*

$$(5.1) \quad \frac{1}{2} g''(x) S^2(x) + \tilde{B}(x) g'(x) + \gamma \left[\mu x - \frac{1 - \gamma}{2} \sigma^2 x^2 - \nu \right] g(x) = 0$$

has the fundamental system

$$(5.2) \quad g_{1,2}(x) = |1/x - 1|^{\rho \mp \Delta} |1 - x|^\gamma$$

where $\Delta \in \mathbf{C}$ is such that $\nu = \frac{\sigma^2 \Delta^2 - \rho^2}{2\gamma}$, on intervals that do not contain the points 0 and 1. In case that $\nu = -\frac{\sigma^2 \rho^2}{2\gamma}$, the equation (5.1) has the following fundamental system

$$(5.3) \quad g_2(x) = g_1(x) \ln \left| \frac{x}{1-x} \right|, \quad \text{where } g_1(x) = \left| \frac{1}{x} - 1 \right|^\rho |1-x|^\gamma$$

on intervals not containing 0 and 1.

Lemma 5.2 Let $\rho_1 < \rho_2$, denote $[l, r] := \{x^2, x \in [\rho_1, \rho_2]\}$. Then the following function

$$\mathcal{D} : D \mapsto \int_{\rho_1}^{\rho_2} \frac{dx}{x^2 - D}$$

is a bijection with domain $\mathbb{R} \setminus [l, r]$ and range $\mathbb{R} \setminus \{0\}$.

Remark 5.3 If $\gamma < 0$, we will employ the following notation $\rho_\alpha := \rho + \gamma\xi_+(\alpha)$, $\rho_\beta := \rho + \gamma\xi_-(\beta)$ and define $D(\alpha, \beta)$ as the unique solution (see lemma 5.2) to the equation

$$\ln \frac{1/\alpha - 1}{1/\beta - 1} = \int_{\rho_\beta}^{\rho_\alpha} \frac{dx}{x^2 - D(\alpha, \beta)}$$

if $\rho_\alpha \neq \rho_\beta$, i.e. $\xi_+(\alpha) \neq \xi_-(\beta)$, and as $\rho_\alpha^2 = \rho_\beta^2$ if $\rho_\alpha = \rho_\beta$, i.e. $\xi_+(\alpha) = \xi_-(\beta)$. Further, we put $u(\alpha, \beta) = (D(\alpha, \beta) - \rho^2)/\gamma$. Then $u(\alpha, \beta)$ is the unique solution to the equation

$$(5.4) \quad \ln \frac{1/\alpha - 1}{1/\beta - 1} = \int_{\xi_-(\beta)}^{\xi_+(\alpha)} \frac{dz}{\gamma z^2 + 2\rho z - u(\alpha, \beta)}$$

if $\xi_+(\alpha) \neq \xi_-(\beta)$ and $u(\alpha, \beta) = \gamma\xi^2 + 2\rho\xi$ if $\xi := \xi_+(\alpha) = \xi_-(\beta)$.

If $\gamma = 0$, we define $u(\alpha, \beta)$ as the unique solution to (5.4) if $\xi_+(\alpha) \neq \xi_-(\beta)$ and $u(\alpha, \beta) := 2\rho\xi$ if $\xi := \xi_+(\alpha) = \xi_-(\beta)$. In case $\gamma = 0$, the right-hand side of (5.4) is equal to

$$\frac{1}{2\rho} \ln \frac{2\rho\xi_+(\alpha) - u(\alpha, \beta)}{2\rho\xi_-(\beta) - u(\alpha, \beta)} \quad \text{if } \rho \neq 0, \quad \text{to} \quad \frac{\xi_-(\beta) - \xi_+(\alpha)}{u(\alpha, \beta)} \quad \text{if } \rho = 0.$$

Hence,

$$(5.5) \quad u(\alpha, \beta) = \frac{\left| \frac{\beta}{1-\beta} \right|^{2\rho} \xi_-(\beta) - \left| \frac{\alpha}{1-\alpha} \right|^{2\rho} \xi_+(\alpha)}{\frac{1}{2\rho} \left[\left| \frac{\beta}{1-\beta} \right|^{2\rho} - \left| \frac{\alpha}{1-\alpha} \right|^{2\rho} \right]} \quad \text{if } \rho \neq 0$$

$$(5.6) \quad = \frac{\xi_-(\beta) - \xi_+(\alpha)}{\ln \left| \frac{\beta}{1-\beta} \right| - \ln \left| \frac{\alpha}{1-\alpha} \right|} \quad \text{if } \rho = 0.$$

Lemma 5.4 Let $\gamma < 0$, then there exists $f \in C^2[\alpha, \beta]$ such that equation (5.1) and the boundary conditions

$$(5.7) \quad g'(\alpha) = \gamma\vartheta_+(\alpha)g(\alpha), \quad g'(\beta) = -\gamma\vartheta_-(\beta)g(\beta)$$

are satisfied with $g(x) := e_\gamma(-f(x))$ and $\nu := \frac{\sigma^2}{2} u(\alpha, \beta)$, where u is given by formula (5.4).

Proof: We will show that there exists $h \in C^2[\alpha, \beta]$ positive on $[\alpha, \beta]$ such that

$$(5.8) \quad g(x) := |x|^{-\rho} |1 - x|^{\rho+\gamma} h(x) / \gamma$$

is a solution to (5.1) and (5.7). By lemma 5.1, the condition (5.1) will be verified if we show that g is a linear combination of g_1, g_2 defined by (5.2) or (5.3), respectively. The condition (5.7) can be rewritten into the condition for the logarithmic derivative of function g

$$\frac{g'(x)}{g(x)} = -\frac{\rho + \gamma x}{x(1-x)} + \frac{h'(x)}{h(x)}$$

in the form $f'(\alpha) = -\frac{1}{\gamma} \frac{g'(\alpha)}{g(\alpha)} = -\vartheta_+(\alpha)$ and $f'(\beta) = -\frac{1}{\gamma} \frac{g'(\beta)}{g(\beta)} = \vartheta_-(\beta)$. These conditions are now of the form

$$(5.9) \quad \frac{h'(\alpha)}{h(\alpha)} = \frac{\rho + \gamma\alpha[1 + \vartheta_+(\alpha)(1-\alpha)]}{\alpha(1-\alpha)} = \frac{\rho_\alpha}{\alpha(1-\alpha)}$$

and $\frac{h'(\beta)}{h(\beta)} = \frac{\rho + \gamma\xi_-(\beta)}{\beta(1-\beta)} = \frac{\rho_\beta}{\beta(1-\beta)}$. Further, denote $D := D(\alpha, \beta)$.

If $\rho_\alpha = \rho_\beta$, define $h(x) := \left|\frac{1}{x} - 1\right|^{-\rho_\alpha} = \left|\frac{1}{x} - 1\right|^{-\rho_\beta}$. If $\rho_\alpha \neq \rho_\beta$ and $D = 0$, then we put $s := \text{sign } \rho_\alpha = \text{sign } \rho_\beta \neq 0$ and

$$(5.10) \quad h(x) := s \cdot \left(\frac{1}{\rho_\alpha} + \ln \frac{1/\alpha - 1}{1/x - 1}\right) = s \cdot \left(\frac{1}{\rho_\beta} + \ln \frac{1/\beta - 1}{1/x - 1}\right).$$

If $D = -a^2 < 0$, we define $h(x) := 2 \sin(\mathcal{A}(x))$, where

$$\mathcal{A}(x) := \text{arccotg} \left(\frac{\rho_\alpha}{a}\right) + a \ln \frac{1/\alpha - 1}{1/x - 1} = \text{arccotg} \left(\frac{\rho_\beta}{a}\right) + a \ln \frac{1/\beta - 1}{1/x - 1}.$$

Further, we denote by $\text{co}A$ the convex hull of a set $A \subseteq \mathbb{R}$. If $\rho_\alpha \neq \rho_\beta$ and $0 < D < x^2$ holds for every $x \in \text{co}\{\rho_\alpha, \rho_\beta\}$, put $\Delta := \sqrt{D} \cdot \text{sign } \rho_\alpha \neq 0$ and define $h(x) := 2 \sinh(\mathcal{B}(x))$, where $\mathcal{B}(x)$ is defined as

$$\text{arctgh} \left(\frac{\rho_\alpha}{\Delta}\right) + \Delta \ln \frac{1/\alpha - 1}{1/x - 1} = \text{arctgh} \left(\frac{\rho_\beta}{\Delta}\right) + \Delta \ln \frac{1/\beta - 1}{1/x - 1}.$$

If $\rho_\alpha \neq \rho_\beta$ and $D > x^2$ holds for every $x \in \text{co}\{\rho_\alpha, \rho_\beta\}$, put $\Delta := \sqrt{D} > 0$ and define $h(x) := 2 \cosh \mathcal{C}(x)$, where

$$\mathcal{C}(x) = \text{arctgh} \left(\frac{\rho_\alpha}{\Delta}\right) + \Delta \ln \frac{1/\alpha - 1}{1/x - 1} = \text{arctgh} \left(\frac{\rho_\beta}{\Delta}\right) + \Delta \ln \frac{1/\beta - 1}{1/x - 1}.$$

In case $\rho_\alpha = \rho_\beta$, the logarithmic derivative of h obviously satisfies (5.9) and function g_1 is by lemma 5.1 a solution to (5.1).

If $\rho_\alpha \neq \rho_\beta$ and $D = 0$, then $0 \notin \text{co}\{\rho_\alpha, \rho_\beta\}$ and therefore $\text{sign } \rho_\alpha = \text{sign } \rho_\beta \neq 0$. By definition of D in this case, $\ln \frac{1/\alpha - 1}{1/\beta - 1} = \int_{\rho_\beta}^{\rho_\alpha} \frac{dx}{x^2} = \frac{1}{\rho_\beta} - \frac{1}{\rho_\alpha}$ and therefore (5.10) correctly defines function h , which obviously satisfies (5.9). It is also obviously a positive function on $[\alpha, \beta]$, since it is monotone and positive at both extreme points α and β . It is also seen that function g defined by (5.8) is a linear combination of g_1 and g_2 , see (5.2) and (5.3).

In case $D = -a^2 < 0$,

$$\ln \frac{1/\alpha - 1}{1/\beta - 1} = \int_{\rho_\beta}^{\rho_\alpha} \frac{dx}{x^2 + a^2} = \frac{1}{a} \left[\text{arccotg} \left(\frac{\rho_\beta}{a}\right) - \text{arccotg} \left(\frac{\rho_\alpha}{a}\right) \right]$$

and therefore function \mathcal{A} is defined correctly. Applying the chain rule, we easily verify that function $h(x) = 2 \sin(\mathcal{A}(x))$ satisfies (5.9). To show that the function h is positive, we need to show that $\mathcal{A}(x) \in (0, \pi)$ holds for every $x \in [\alpha, \beta]$. Since $\mathcal{A}(x)$ is a monotone function, we have to verify only that $\mathcal{A}(\alpha), \mathcal{A}(\beta) \in (0, \pi)$, which is obviously satisfied as $\operatorname{arccotg}$ attains values only in $(0, \pi)$. Function g defined by (5.8) is a (complex) linear combination of functions g_1 and g_2 with $\Delta := ia$ if and only if h is a (complex) linear combination of functions $|1/x - 1|^{\pm ia} = \cos(a \ln |1/x - 1|) \pm i \sin(a \ln |1/x - 1|)$ which obviously is, by formula $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

In case $\rho_\alpha \neq \rho_\beta$ and $D = \Delta^2 > 0$, $\ln \frac{1/\alpha - 1}{1/\beta - 1}$ is equal to

$$\int_{\rho_\beta}^{\rho_\alpha} \frac{dx}{x^2 - \Delta^2} = \frac{1}{\Delta} \left[\operatorname{arccotgh} \left(\frac{\rho_\beta}{\Delta} \right) - \operatorname{arccotgh} \left(\frac{\rho_\alpha}{\Delta} \right) \right]$$

if $x^2 > \Delta^2$ for every $x \in \operatorname{co}\{\rho_\alpha, \rho_\beta\}$ and to

$$\int_{\rho_\beta}^{\rho_\alpha} \frac{dx}{x^2 - \Delta^2} = \frac{1}{\Delta} \left[\operatorname{arctgh} \left(\frac{\rho_\beta}{\Delta} \right) - \operatorname{arctgh} \left(\frac{\rho_\alpha}{\Delta} \right) \right]$$

if $x^2 < \Delta^2$ for every $x \in \operatorname{co}\{\rho_\alpha, \rho_\beta\}$. Therefore, the functions \mathcal{B} or \mathcal{C} are defined correctly in these cases, respectively. Again by the chain rule, we easily obtain that h satisfies (5.9). In case that $D > x^2$ for every $x \in \operatorname{co}\{\rho_\alpha, \rho_\beta\}$, h is defined as $2 \cosh(\mathcal{C}(x))$, which is obviously a positive function. In case that $D < x^2$ for every $x \in \operatorname{co}\{\rho_\alpha, \rho_\beta\}$, h is defined as $2 \sinh(\mathcal{B}(x))$. Hence, $h(x)$ is positive if and only if $\mathcal{B}(x)$ is positive. Since $\mathcal{B}(x)$ is a monotone function, we only need to show that $\mathcal{B}(\alpha), \mathcal{B}(\beta) > 0$. This condition is satisfied, since the function $\operatorname{arccotgh}$ is positive on $(0, \infty)$ and Δ is defined so that $\frac{\rho_\alpha}{\Delta}, \frac{\rho_\beta}{\Delta} > 0$. We have used $\operatorname{sign} \rho_\alpha = \operatorname{sign} \rho_\beta$, which follows from the condition that $0 < D < x^2$ for every $x \in \operatorname{co}\{\rho_\alpha, \rho_\beta\}$, i.e. $0 \notin \operatorname{co}\{\rho_\alpha, \rho_\beta\}$. By definition of \sinh and \cosh , function h is a linear combination of functions $\exp\{\pm \Delta \ln |1/x - 1|\} = |1/x - 1|^{\pm \Delta}$ and therefore function g defined by (5.8) is a linear combination of functions g_1 and g_2 . \square

Remark 5.5 (a) The function $h_\infty(x) := \frac{1}{1-x}$ is a solution to

$$(5.11) \quad B(x)h(x) + \frac{1}{2} S^2(x)h'(x) = \mu x - \frac{1}{2} \sigma^2 x^2 - \nu$$

with $\nu := 0$ on intervals not containing 0 and 1. (b) The function

$$h(x) = \left| \frac{1}{x} - 1 \right|^{2u} \frac{1}{x(1-x)}$$

satisfies $\frac{1}{2}h'(x)S^2(x) + h(x)B(x) = \sigma^2(\rho - u) \left| \frac{1}{x} - 1 \right|^{2u}$ on intervals not containing 0 and 1. In particular, the function

$$h_0(x) = \left| \frac{1}{x} - 1 \right|^{2\rho} \frac{1}{x(1-x)}$$

is a solution to (5.11) with the right-hand side zero. (c) Further,

$$h_1(x) := \frac{k(x)}{x(1-x)}, \quad \text{where} \quad k(x) = \frac{1}{2\rho} \quad \text{if} \quad \rho \neq 0$$

$$k(x) = \ln \left| \frac{x}{1-x} \right| \quad \text{if} \quad \rho = 0$$

is a solution to (5.11) with the right-hand side $\frac{\sigma^2}{2}$.

Lemma 5.6 *Let $a_0 \in \mathbf{R}$ and $\nu + \frac{\sigma^2}{2} a_1 = 0$, then*

$$h(x) := a_0 h_0(x) + a_1 h_1(x) + h_\infty(x)$$

is a solution to (5.11). Moreover, if

$$a_0 = -\frac{\xi_-(\beta) - \xi_+(\alpha)}{\left| \frac{1}{\beta} - 1 \right|^{2\rho} - \left| \frac{1}{\alpha} - 1 \right|^{2\rho}}, \quad a_1 = -\frac{\left| \frac{\beta}{1-\beta} \right|^{2\rho} \xi_-(\beta) - \left| \frac{\alpha}{1-\alpha} \right|^{2\rho} \xi_+(\alpha)}{\frac{1}{2\rho} \left[\left| \frac{\beta}{1-\beta} \right|^{2\rho} - \left| \frac{\alpha}{1-\alpha} \right|^{2\rho} \right]}$$

in case that $\rho \neq 0$ and

$$a_0 = \frac{\xi_-(\beta) \ln \left| \frac{\alpha}{1-\alpha} \right| - \xi_+(\alpha) \ln \left| \frac{\beta}{1-\beta} \right|}{\ln \left| \frac{\beta}{1-\beta} \right| - \ln \left| \frac{\alpha}{1-\alpha} \right|}, \quad a_1 = -\frac{\xi_-(\beta) - \xi_+(\alpha)}{\ln \left| \frac{\beta}{1-\beta} \right| - \ln \left| \frac{\alpha}{1-\alpha} \right|},$$

in case $\rho = 0$, respectively, then h also satisfies the boundary conditions $h(\alpha) = -\vartheta_+(\alpha)$, $h(\beta) = \vartheta_-(\beta)$.

In particular, there exists $f = \int h(x) dx \in C^2[\alpha, \beta]$ such that

$$(5.12) \quad \tilde{B}(x)f'(x) + \frac{1}{2}S^2(x)[f''(x) - \gamma f'(x)^2] = \mu x - \frac{1}{2}\sigma^2(1-\gamma)x^2 - \nu$$

and

$$(5.13) \quad f'(\alpha) = -\vartheta_+(\alpha), \quad f'(\beta) = \vartheta_-(\beta).$$

hold in case that $\nu = \frac{\sigma^2}{2} u(\alpha, \beta)$.

Proof: The first part of the statement immediately follows from remark 5.5. The boundary conditions hold if and only if

$$\begin{pmatrix} h_0(\alpha) & h_1(\alpha) \\ h_0(\beta) & h_1(\beta) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \begin{pmatrix} \frac{1}{1-\alpha} \\ \frac{1}{1-\beta} \end{pmatrix} = \begin{pmatrix} -\frac{b}{1+b\alpha} \\ \frac{c}{1-c\beta} \end{pmatrix}.$$

Multiplying the first row by $\alpha(1 - \alpha)$ and the second one by $\beta(1 - \beta)$, we obtain

$$\begin{pmatrix} \left|\frac{1}{\alpha} - 1\right|^{2\rho} & k(\alpha) \\ \left|\frac{1}{\beta} - 1\right|^{2\rho} & k(\beta) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -\xi_+(\alpha) \\ -\xi_-(\beta) \end{pmatrix}.$$

Therefore the boundary conditions are satisfied if and only if

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{\mathcal{D}} \begin{pmatrix} k(\beta) & -k(\alpha) \\ -\left|\frac{1}{\beta} - 1\right|^{2\rho} & \left|\frac{1}{\alpha} - 1\right|^{2\rho} \end{pmatrix} \begin{pmatrix} -\xi_+(\alpha) \\ -\xi_-(\beta) \end{pmatrix},$$

where $\mathcal{D} := \left|\frac{1}{\alpha} - 1\right|^{2\rho} k(\beta) - \left|\frac{1}{\beta} - 1\right|^{2\rho} k(\alpha)$. The remaining part of the statement follows from remark 5.3, which says that $u(\alpha, \beta) = -a_1$. \square

Theorem 5.7 *Suppose that the strategy $[(\alpha, \beta)]$ is applied, then*

$$\nu(\alpha, \beta) := \lim_{t \rightarrow \infty} \frac{1}{t} e_\gamma^{-1} E e_\gamma(\ln Y_t) = \frac{\sigma^2}{2} u(\alpha, \beta),$$

where $u(\alpha, \beta)$ is defined by (5.4). Moreover, $EY_t^\delta < \infty$ holds for every $\delta < 0$.

Proof. By lemma 5.2, $u(\alpha, \beta)$ is defined well by (5.4). By lemma 5.4 there exists $f \in C^2[\alpha, \beta]$ such that (5.12) and (5.13) hold with $\nu = \frac{\sigma^2}{2} u(\alpha, \beta)$ in case $\gamma < 0$. By lemma 5.6, such a function f exists in case $\gamma = 0$. By theorem 4.1, $\nu(\alpha, \beta) = \nu$, which is equal to $\frac{\sigma^2}{2} u(\alpha, \beta)$ and $e_\gamma(\ln Y_t - f(G_t) - \nu t)$ is an \mathcal{F}_t -martingale. In particular, if $\gamma < 0$, and $t \geq 0$, we get that

$$\begin{aligned} EY_t^\gamma &= \gamma E e_\gamma(\ln Y_t) \leq \gamma e^{\gamma(\nu t + \hat{m})} E e_\gamma(\ln Y_t - f(G_t) - \nu t) \\ &= \gamma e^{\gamma(\nu t + \hat{m})} E e_\gamma(\ln y_0 - f(G_0)) \leq e^{\gamma(\nu t + \hat{m} - \check{m})} y_0^\gamma < \infty, \end{aligned}$$

where

$$\hat{m} := \min\{f(x); x \in [\alpha, \beta]\}, \quad \check{m} := \max\{f(x); x \in [\alpha, \beta]\}.$$

Since we only assume that $\alpha < \beta$ are such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and that $\gamma < 0$, we immediately obtain that the more-over part of the statement holds. It is sufficient to put $\gamma := \delta < 0$ and to apply the first part of the statement. \square

6. PROPERTIES OF FUNCTION u

In this section, we are going to show that there exist $\alpha < \beta, \nu \in \mathbb{R}, f \in C^2(-1/b, 1/c)$ such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and such that assumptions of theorem 4.4 are satisfied.

Lemma 6.1 *Let $\alpha < \beta$ and $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$, let $\omega_\gamma \in (0, |\theta_\gamma| \wedge |1 - \theta_\gamma|)$ be such that*

$$(6.1) \quad \xi_+(\alpha) = \theta_\gamma - \omega_\gamma, \quad \xi_-(\beta) = \theta_\gamma + \omega_\gamma, \quad u(\alpha, \beta) = (1 - \gamma)(\theta_\gamma^2 - \omega_\gamma^2).$$

Then $u(\alpha, x) \neq u(\alpha, \beta) \neq u(x, \beta)$ hold for every $x \in (\alpha, \beta)$.

Proof. Let $x \in (\alpha, \beta)$. We are going to show that $u(\alpha, x) \neq u(\alpha, \beta)$. The proof of the remaining part of the statement would be similar. First, we show that $\xi_-(x) \geq \xi_+(\alpha)$. If $-1/b < \alpha < \beta < 0$ or if $1 < \alpha < \beta < 1/c$, then it follows from the following inequalities $\xi_+(\alpha) < \alpha < x < \xi_-(x)$. If $0 < \alpha < \beta < 1$, then we obtain from the definition of $u(\alpha, \beta)$ that

$$(6.2) \quad 0 > -\ln \frac{1/\alpha - 1}{1/\beta - 1} = \int_{\xi_+(\alpha)}^{\xi_-(\beta)} \frac{dz}{\gamma z^2 + 2\rho z - u(\alpha, \beta)}.$$

Since the right-hand side has to be defined correctly and finite, we get that the function $\gamma z^2 + 2\rho z - u(\alpha, \beta)$ does not change the sign on $[\xi_+(\alpha), \xi_-(\beta)]$. Then it follows from (6.2) and from the inequality $\xi_+(\alpha) = \theta_\gamma - \omega_\gamma < \theta_\gamma + \omega_\gamma = \xi_-(\beta)$ that $u(\alpha, \beta) > \gamma z^2 + 2\rho z$ holds for every $z \in [\xi_+(\alpha), \xi_-(\beta)]$. We obtain from the definition of $u(\alpha, x)$ that (6.2) holds also when β is replaced by x . Since the integrand cannot change the sign between $\xi_+(\alpha)$ and $\xi_-(x)$ and it is negative at $\xi_+(\alpha)$, we obtain from (6.2) with β replaced by x that $\xi_-(x) \geq \xi_+(\alpha)$. It follows from the definition of ξ_- that $\xi_-(x) < \xi_-(\beta)$, since $x < \beta < 1/c$.

Now, we are going to show that the case $u(\alpha, x) = u(\alpha, \beta)$ leads to a contradiction. Let $u(\alpha, x) = u(\alpha, \beta)$, then the definition of $u(\alpha, x)$ and $u(\alpha, \beta)$ gives us that

$$(6.3) \quad -\ln \frac{1+b}{1-c} = \int_{\xi_+(\alpha)}^{\xi_-(\beta)} \lambda(z) dz = \int_{\xi_+(\alpha)}^{\xi_-(x)} \lambda(z) dz,$$

where $\lambda(z)$ is defined as

$$\frac{1}{z(1-z)} + \frac{1}{\gamma z^2 + 2\rho z - u(\alpha, \beta)} = \frac{(1-\gamma)[\omega_\gamma^2 - (\theta_\gamma - z)^2]}{z(1-z)[\gamma z^2 + 2\rho z - u(\alpha, \beta)]}.$$

In particular, $\int_{\xi_-(x)}^{\xi_-(\beta)} \lambda(z) dz = 0$. Since this integral has to be well defined and finite, the function $\lambda(z)$ cannot have a pole on $[\xi_-(x), \xi_-(\beta)]$. Obviously, λ is a continuous function on a non-degenerate interval $[\xi_-(x), \xi_-(\beta)]$ with the mean value zero. Hence, there exists $z \in (\xi_-(x), \xi_-(\beta))$ such that $\lambda(z) = 0$, but it follows from the definition of λ that $\lambda(z) \neq 0$ holds for every $z \in (\theta_\gamma - \omega_\gamma, \theta_\gamma + \omega_\gamma) \supseteq (\xi_-(x), \xi_-(\beta))$. \square

Lemma 6.2 *Let $\alpha < \beta$ and $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$, let $\omega_\gamma \in (0, |\theta_\gamma| \wedge |1 - \theta_\gamma|)$ be such that (6.1) is satisfied. Then there exists $f \in C^2(-1/b, 1/c)$ such that assumptions of theorem 4.4 are satisfied with $\nu = \frac{1-\gamma}{2} \sigma^2 (\theta_\gamma^2 - \omega_\gamma^2)$.*

Proof. We are going to find $f \in C^2(-1/b, 1/c)$ such that (4.1) hold for every $y \in (-1/b, \alpha]$, $z \in [\beta, 1/c)$ and $x \in (\alpha, \beta)$. Then the last equality in (4.1) will hold also for $x \in \{\alpha, \beta\}$. By lemma 5.6, there exists $f \in C^2[\alpha, \beta]$ satisfying (5.12) and (5.13) in case $\gamma = 0$. By lemma 5.4, such a function exists also if $\gamma < 0$. We obtain from the boundary conditions (5.13) that we can extend f to $(-1/b, 1/c)$ so that

$$\delta_+^f(y) = \delta_-^f(z) = 0, \quad \text{i.e.} \quad f'(y) = -\vartheta_+(y), \quad f'(z) = \vartheta_-(z)$$

hold for every $y \in (-1/b, \alpha]$ and $z \in [\beta, 1/c)$. Obviously, (5.12) is satisfied for every $x \in (\alpha, \beta)$ and therefore $d_f'' = 0$ holds on (α, β) . To show that $f \in C^2(-1/b, 1/c)$ means to show that $f''(\alpha_+) = f'(\alpha)^2$ and $f''(\beta_-) = f'(\beta)^2$, since $f''(\alpha_-) = f'(\alpha)^2$ and $f''(\beta_+) = f'(\beta)^2$. We are going to show the first equality, the second one could be obtained similarly.

Since $S^2(\alpha) \neq 0$, we are only to show that (5.12) holds if we replace $f'(x)$ by $f'(\alpha) = -\vartheta_+(\alpha)$ and $f''(x)$ by $-\vartheta_+'(\alpha) = \vartheta_+(\alpha)^2$. Since $f''(x) = f'(x)^2$ in that case, we obtain (5.12) in the form that the right-hand side of (4.3) has to be zero, but it follows from the assumptions, since

$$\nu = \frac{1-\gamma}{2} \sigma^2(\theta_\gamma^2 - \omega_\gamma^2), \quad [\theta_\gamma - x(1 - (1-x)f'(x))]^2 = [\theta_\gamma - \xi_+(\alpha)]^2 = \omega_\gamma^2.$$

Obviously, the condition $\xi_+(\alpha) < \theta_\gamma < \xi_-(\beta)$ is satisfied, since $\omega_\gamma > 0$ and $u(\alpha, x) \neq u(\alpha, \beta) \neq u(x, \beta)$ hold for every $x \in (\alpha, \beta)$ by lemma 6.1 and theorem 5.7. Further, we obtain from theorem 5.7 the same inequalities with u replaced by ν . By lemma 4.3, (4.2) hold for every $y \in (\alpha, 1/c)$, $z \in (-1/b, \beta)$ and $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$. \square

Lemma 6.3 *Let $\theta_\gamma \notin \{0, 1\}$, then*

$$(6.4) \quad \mathcal{I} : \omega_\gamma \mapsto \int_{\theta_\gamma - \omega_\gamma}^{\theta_\gamma + \omega_\gamma} \frac{(1-\gamma)[\omega_\gamma^2 - (\theta_\gamma - x)^2]}{x(1-x)[\gamma x^2 + 2\rho x - (1-\gamma)(\theta_\gamma^2 - \omega_\gamma^2)]} dx$$

is a continuous decreasing function on $(0, \tilde{\omega}_\gamma)$ and it attains all negative values on this interval, where $\tilde{\omega}_\gamma := |\theta_\gamma| \wedge |1 - \theta_\gamma|$ if $\theta_\gamma \notin [0, 1]$ and

$$\tilde{\omega}_\gamma := |\theta_\gamma| \wedge |1 - \theta_\gamma| \wedge \sup\{\omega_\gamma \geq 0 : \sup \mathbb{D}_{\omega_\gamma} < 0\} > 0$$

if $\theta_\gamma \in (0, 1)$, where

$$\mathbb{D}_{\omega_\gamma} : x \in [\theta_\gamma - \omega_\gamma, \theta_\gamma + \omega_\gamma] \mapsto \gamma x^2 + 2\rho x - (1-\gamma)(\theta_\gamma^2 - \omega_\gamma^2)$$

is positive in case $\theta_\gamma \notin [0, 1]$ and it is negative in case $\theta_\gamma \in (0, 1)$ provided that $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$.

Proof. The function $\mathbb{D}_{\omega_\gamma}$ is obviously concave and it attains the value $-(\theta_\gamma \pm \omega_\gamma)(1 - \theta_\gamma \mp \omega_\gamma)$ at the extreme points $\theta_\gamma \pm \omega_\gamma$. If $\theta_\gamma \notin [0, 1]$, such values are positive and therefore $\mathbb{D}_{\omega_\gamma}(x) > 0$ holds for every $x \in [\theta_\gamma - \omega_\gamma, \theta_\gamma + \omega_\gamma]$. In particular, $\mathcal{I}(\omega_\gamma)$ is defined correctly by (6.4).

Let $\theta_\gamma \in (0, 1)$. Obviously, $\omega_\gamma \mapsto \sup \mathbb{D}_{\omega_\gamma}$ is a continuous increasing function with the value $\theta_\gamma[\theta_\gamma - 1] < 0$ at zero and therefore $\tilde{\omega}_\gamma > 0$. Since $\mathbb{D}_{\omega_\gamma}(x) < 0$ holds by definition of $\tilde{\omega}_\gamma$ for every x and ω_γ such that $|\theta_\gamma - x| < \omega_\gamma < \tilde{\omega}_\gamma$, we get that the integrand in (6.4) is a continuous function on $[\theta_\gamma - \omega_\gamma, \theta_\gamma + \omega_\gamma]$ and therefore $\mathcal{I}(\omega_\gamma)$ is well defined by (6.4) on $[0, \tilde{\omega}_\gamma)$.

Now, we are going to show that $\mathcal{I}(\omega_\gamma)$ is a continuous decreasing function on $(0, \tilde{\omega}_\gamma)$. Obviously, the integrand in (6.4) attains the value zero at both extreme points $\theta_\gamma \pm \omega_\gamma$ and therefore

$$\frac{d}{d\omega_\gamma} \mathcal{I}(\omega_\gamma) = \int_{\theta_\gamma - \omega_\gamma}^{\theta_\gamma + \omega_\gamma} \frac{d}{d\omega_\gamma} \frac{dx}{\mathbb{D}_{\omega_\gamma}(x)} = - \int_{\theta_\gamma - \omega_\gamma}^{\theta_\gamma + \omega_\gamma} \frac{2\omega_\gamma(1 - \gamma) dx}{\mathbb{D}_{\omega_\gamma}(x)^2} < 0$$

whenever $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$. Obviously, $\mathcal{I}(0) = 0$ and \mathcal{I} is continuous at zero. We are to show that $\mathcal{I}(\omega_\gamma) \rightarrow -\infty$ as $\omega_\gamma \uparrow \tilde{\omega}_\gamma$, but it follows from Fatou's lemma, since the integrand in (6.4) is always non-positive and the limiting integrand is not integrable. \square

Lemma 6.4 *Let $\theta_\gamma \notin \{0, 1\}$ and $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$ be such that $\mathcal{I}(\omega_\gamma) = -\ln \frac{1+b}{1-c}$. Then there exist $\alpha < \beta$ such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and $\xi_+(\alpha) = \theta_\gamma - \omega_\gamma, \xi_-(\beta) = \theta_\gamma + \omega_\gamma$.*

Proof. If $\theta_\gamma \in (0, 1)$, then $\theta_\gamma \pm \omega_\gamma \in (0, 1)$ and therefore

$$\alpha := \xi_+^{-1}(\theta_\gamma - \omega_\gamma) \in (0, 1) \ni \xi_-^{-1}(\theta_\gamma + \omega_\gamma) =: \beta.$$

Obviously, $\alpha < \xi_+(\alpha) = \theta_\gamma - \omega_\gamma < \theta_\gamma + \omega_\gamma = \xi_-(\beta) < \beta$.

Now, we are going to show that there exist $-1/b < \alpha < \beta < 0$ satisfying (6.1) in case $\theta_\gamma < 0$. The proof of existence $1 < \alpha < \beta < 1/c$ satisfying (6.1) in case $\theta_\gamma > 1$ would be similar. Let $\theta_\gamma < 0$. Then $\theta_\gamma - \omega_\gamma < 0$ and therefore $\alpha := \xi_+^{-1}(\theta_\gamma - \omega_\gamma) < 0$. Since $\mathbb{D}_{\omega_\gamma}(x) > 0$ holds for every $x \in [\theta_\gamma - \omega_\gamma, \theta_\gamma + \omega_\gamma]$ by lemma 6.3, we obtain

$$(6.5) \quad \ln \frac{1 + b \frac{\theta_\gamma + \omega_\gamma}{1 - c \frac{\theta_\gamma - \omega_\gamma}{1 - \theta_\gamma - \omega_\gamma}}}{1 - c \frac{\theta_\gamma - \omega_\gamma}{1 - \theta_\gamma - \omega_\gamma}} = - \int_{\theta_\gamma - \omega_\gamma}^{\theta_\gamma + \omega_\gamma} \frac{dx}{\mathbb{D}_{\omega_\gamma}(x)} < 0.$$

The left-hand side is a sum of $\ln(1 + b) + \ln \left| \frac{1 - \theta_\gamma + \omega_\gamma}{\theta_\gamma - \omega_\gamma} \right| = \ln |1/\alpha - 1| > 0$ and $-\ln(1 - c) - \ln \left| \frac{1 - \theta_\gamma - \omega_\gamma}{\theta_\gamma + \omega_\gamma} \right|$ which has to be negative. In particular, $\theta_\gamma + \omega_\gamma \neq \frac{1-c}{-c}$ and therefore we can put $\beta := \xi_-^{-1}(\theta_\gamma + \omega_\gamma)$. Further, we obtain from the definition of ξ_- that

$$(6.6) \quad \ln |1/\beta - 1| = \ln(1 - c) + \ln \left| \frac{1 - \theta_\gamma - \omega_\gamma}{\theta_\gamma + \omega_\gamma} \right| > 0.$$

Since $\theta_\gamma + \omega_\gamma \notin [0, 1]$, we get that $\beta = \xi_-^{-1}(\theta_\gamma + \omega_\gamma) \notin [0, 1]$ and we obtain from (6.6) that $\beta < 0$. Since $\alpha, \beta < 0$, the inequality $\alpha < \beta$ is equivalent to the inequality $\ln \left| \frac{1/\alpha - 1}{1/\beta - 1} \right| < 0$ and it holds, since its left-hand side is equal to the left-hand side of (6.5) by the definition of ξ_+ and ξ_- . \square

Corollary 6.5 *Let $\theta_\gamma \notin \{0, 1\}$. Then there exists just one $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$ such that $\mathcal{I}(\omega_\gamma) + \ln \frac{1+b}{1-c} = 0$. Further,*

$$(6.7) \quad -1/b < \alpha := \xi_+^{-1}(\theta_\gamma - \omega_\gamma) < \xi_-^{-1}(\theta_\gamma + \omega_\gamma) =: \beta < 1/c$$

are such that $0, 1 \notin [\alpha, \beta]$ and there exists $f \in C^2(-1/b, 1/c)$ such that the assumptions of theorem 4.4 are satisfied with $\nu = \frac{1-\gamma}{2} \sigma^2 (\theta_\gamma^2 - \omega_\gamma^2)$.

Proof. By lemma 6.3, there exists just one $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$ such that $\mathcal{I}(\omega_\gamma) + \ln \frac{1+b}{1-c} = 0$. By lemma 6.4, (6.7) and $0, 1 \notin [\alpha, \beta]$ hold if α and β are defined by (6.7). We obtain from the equality $\mathcal{I}(\omega_\gamma) + \ln \frac{1+b}{1-c} = 0$ and the definition of u that $u(\alpha, \beta) = (1-\gamma)(\theta_\gamma^2 - \omega_\gamma^2)$. Hence, (6.1) are satisfied and we obtain from lemma 6.2 that there exists $f \in C^2(-1/b, 1/c)$ such that assumptions of theorem 4.4 are satisfied. \square

7. TIME CHANGE AND NON-ZERO INTEREST RATE

Remark 7.1 Let X_t be a one-dimensional \mathcal{F}_t -geometric Brownian motion (2.1) with $\mu \in \mathbb{R}, \Sigma^{1/2} = \sigma > 0$ and $x > 0$. By corollary 6.5, there exists $\omega_\gamma \in (0, \tilde{\omega}_\gamma)$ and $\alpha < \beta$ such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ and $f \in C^2(-1/b, 1/c)$ satisfying assumptions of theorem 4.4 with $\nu = \frac{1-\gamma}{2} \sigma^2 (\theta_\gamma^2 - \omega_\gamma^2)$. Let $(\tau(s), s \geq 0)$ be a non-decreasing continuous system of \mathcal{F}_t -stopping times such that $\tau(0) = 0$ and $\tau(s) \rightarrow \infty$ as $t \rightarrow \infty$ and let $\tilde{X}_s := X_{\tau(s)}$ be the stock market price at time s .

a) Let H_t^+, H_t^- be the control processes corresponding to the strategy $[(\alpha, \beta)]$ and let Y_t denote the corresponding portfolio market price in the model with stock market price X_t at time t . Let us consider the strategy with control processes $\tilde{H}_s := H_{\tau(s)}^+$ and $\tilde{H}_s := H_{\tau(s)}^-$ in the time-changed model. By lemma 3.1 applied to the processes with the time argument t , we obtain after substitution that

$$\tilde{Y}_s := Y_{\tau(s)} = y_0 + \int_0^s \tilde{H}_u d\tilde{X}_u - b\tilde{X}_u d\tilde{H}_u^+ - c\tilde{X}_u d\tilde{H}_u^-$$

and therefore \tilde{Y}_s can be interpreted as the portfolio market price at time s in the time-changed model and theorem 4.4 gives that the following equalities hold almost surely

$$\nu = \lim_{t \rightarrow \infty} \frac{1}{t} \ln Y_t = \lim_{s \rightarrow \infty} \frac{1}{\tau(s)} \ln \tilde{Y}_s.$$

b) We are going to show that (7.1) holds almost surely for a wide class of strategies. Let us consider a strategy with the portfolio market price $\tilde{Y}_s \in (0, \infty)$ and that $\tilde{G}_s := \tilde{H}_s \tilde{X}_s \tilde{Y}_s^{-1} \in [\pi_+, \pi_-] \subseteq (-1/b, 1/c)$ hold for every $s \geq 0$ almost surely, where $\tilde{H}_s := H_0 + \tilde{H}_s^+ - \tilde{H}_s^-$ and where $\tilde{H}_s^+, \tilde{H}_s^-$ are $\mathcal{F}_{\tau(s)}$ -adapted control processes. Further, denote $\varphi(t) := \inf\{s \geq 0 : \tau(s) \geq t\}$ and assume that $E\tilde{Y}_{\varphi(t)}^\delta < \infty$ holds for every $t \geq 0$ and $\delta < 0$. This assumption is satisfied for example if there exists $\varepsilon \in (0, 1)$ such that $\tilde{Y}(s) \geq \varepsilon y_0$ holds for every $s \geq 0$

almost surely. Since $\tilde{H}_s^\pm, \varphi(t)$ are non-decreasing and left-continuous processes, we obtain that $H_t^\pm := \tilde{H}_{\varphi(t)}^\pm$ have the same property. They are also \mathcal{F}_t -measurable. By lemma 3.1 applied to the processes with the time argument s and the filtration $\mathcal{F}_{\tau(s)}$, we obtain that

$$Y_t := y_0 + \int_0^t H_u dX_u - bX_u dH_u^+ - cX_u dH_u^- \geq \tilde{Y}_s$$

provided that $t = \tau(s)$. Then we obtain from theorem 4.4 that the following inequalities hold almost surely

$$(7.1) \quad \limsup_{s \rightarrow \infty} \frac{1}{\tau(s)} \ln \tilde{Y}_s \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t \leq \nu.$$

Remark 7.2 We have considered a money market only with zero interest rate. In case of positive interest rate r , we denote by Z_t the stock market price at time t and by $X_t = e^{-rt}Z_t$ the discounted stock market price. If Z_t is a geometric Brownian motion $dZ_t = Z_t\tilde{\mu}dt + Z_t\sigma dW_t$, then X_t is also a geometric Brownian motion with a stochastic differential (2.1), where $\mu := \tilde{\mu} - r$. Now, it is sufficient to define Y_t as a discounted portfolio market price at time t and to apply the above mentioned results to obtain the optimal strategy for this case.

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