An axiomatic basis of accounting: a structuralist reconstruction

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ABSTRACT. Set-theoretic axiomatizations are given for a model of accounting with double classification, and a general core-model for accounting. The empirical status, and "representational" role of systems of accounts, as well as the problem of how to assign "correct" values to the goods accounted, are analyzed in precise terms. A net of special laws based on the core-model is described.

Keywords: Accounting, axiomatic model, theory, theory-net, valuation-problem.

I. INTRODUCTION

During the last few decades many attempts to axiomatize accounting have been undertaken, but ours seems to be the first collaboration in this area between a philosopher of science (of structuralist background) and an accounting theorist. We hope the reconstruction to yield a viable way of catching the *essence and basic structure of accounting* as rigorously as possible; furthermore, it offers a 'successful application' of structuralist metatheory, putting special emphasis on the *semantic* relationship between empirical data and conceptual representation. Finally, a picture of this structure should afford deeper insight into the many aspects of accounting, particularly into its epistemological status and the structure of its physical and socio-economic dualities – of which the double-entry is merely a technical consequence. And although our formalization – being primarily an intellectual exercise – is not likely to have immediate impact on the thinking of practicing accountants,[2] its potential significance for computer application and information systems theory should not be underrated. It is no coincidence that the first attempt of mathematical axiomatization of accounting immediately led into the field of financial and budgetary simulation – see Mattessich (1961, 1964/79).

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Our reconstruction shows that accounting has the same overall structure as have other *empirical* theories, namely the form of a theory-net consisting of a core model and a net of specializations of this model. As in other cases, the core model itself is empirically empty. But the *physical duality*, which arises out of the transfer of a commodity from one entity to another, as well as the *socio-economic dualities*, which reflect either investment – ownership relations or borrower–lender relations, are all empirical phenomena; they become *normative* when interpreted from the point of view of *accountability*: every economic output has to be empirically accounted for in terms of every corresponding economic input or *vice versa* – even if the latter is to some extent consumption (e.g. intermediate consumption as 'costs' and final consumption as 'dividends'). Thus a *symmetry* emerges that is not unlike the one arising from the conservation principle of energy in physics: all energy output is accounted for in terms of the energy inputs – even if some energy has become useless or dissipated. Such assumptions as 'every input equals its corresponding output' may be considered tautological, but how much of this input (or output) is capital formation, how much is income, how much is income distribution, etc., are empirical informations. Thus it would be incorrect to believe that the basis of accounting is purely analytical, imposing nothing but a debit–credit tautology upon economic reality.

II. THE EVOLUTION OF FORMALIZED ACCOUNTING THEORY

> "Axiomatics does not burst upon the scene unprepared. There will have been a vast amount of preparatory exploration and thinking, much of it tentative and in parts. Some will have been in mathematical form, some not." O. Morgenstern (1963), p. 24.

The earliest, purely verbal attempt of formulating accounting 'postulates' can be found by Paton (1922). Decades later Mattessich (1957 and 1964/77)[3] presented his matrix-algebraic and set-theoretical axiomatizations of accounting together with an application to financial simulation (see Mattessich, 1961 and 1964/79) which ultimately led to the present microcomputer spreadsheet programmes available in VISI-CALC, SUPER-CALC, LOTUS 1-2-3, and other best-selling software. Since this time, many attempts to axiomatize accounting have been undertaken in America, Argentina, Australia, Canada, England, Germany, Italy, and Japan: e.g. Winborn (1962), Ijiri (1965, 1967, 1975, 1979, and 1989), Kosiol (1970), Schweitzer (1970), Saito (1972, 1973) with its response by Mattessich (1973), Galassi (1978), Orbach (1978), Tippet (1978), Carlson and Lamb (1981), Tanaka (1982), Deguchi and Nakano (1986), Ávila, Bravo and Scarano (1988), Nehmer (1988), DePree (1989) and, above all, a series of papers by Willett (1987, 1988, 1989). To this have to be added numerous *non-mathematical* formulations of accounting postulates, from Moonitz (1961/82), Chambers (1966), and the American Accounting Association's (1966) *A Statement of Basic Accounting Theory* to the more recent search for a Conceptual Framework by the Financial Accounting Standards Board (1976, 1978–80) of the U.S.A. – cf. also Zeff (1982).

The differences between various axiomatic systems are primarily due to a difference in conceptual (mathematical) apparatus and a different choice of undefined notions. Both of these problems hold for all axiomatic systems (whether of mathematics, accounting or any other kind), since the choice between the many conceptual structures available, as well as that of the 'primitives', are both a matter of taste. In order to overcome this trend toward individualism and diversity, a strong incentive is required. In this paper we offer such an incentive in the form of 'epistemic structuralism' or 'neostructuralism' which has already undertaken the axiomatization of a series of empirical theories in other areas.

Apart from the more rigorous formulation made possible by means of set-theoretical predicates, we here do not deal with a pure but an *applied* or teleologic science. This is manifested in the various 'interpretations', or in structuralistic lingo, in the various 'specializations' of our accounting theory-net, the formulation of which now depends on a specification of the relevant information objective (e.g. *physical* capital maintenance vs. *financial* capital maintenance, and *nominal* capital maintenance vs. *real* capital maintenance). In other words, the various specializations, which in a pure science serve a more detailed description, might be used in an applied science to differentiate between different goal assumptions and their consequences. Thus ours seems to be one of the first axiomatization attempts for an *applied* science. But it must be pointed out that here only the *input–output basis* is involved, and not an expansion into the many behavioral aspects nowadays modeled in finance theory or in information-economics and the agency theory of accounting (for an overview see Mattessich, 1984/89) – these are separate areas of formalization, and their integration into an axiomatic framework belongs to the development of the 'theory nets', and must be postponed – yet a partial sketch of such networks has been presented in Mattessich (1987a).

III. THE ACCOUNTING DATA BASIS

The empirical events represented by accounting since prehistoric and ancient times[4] are based on data about quantities and values of *economic objects*, here called *e-objects* (physical assets as well as ownership claims and debt claims) which are transferred between and within 'holders' $h \in H$ who may engage in such economic activities as buying and selling, producing, consuming and distributing goods or services, owing or owning debts, (i.e. lending or borrowing funds), investing in equities, controlling, recording and keeping track of assets, etc. The set O of all economic objects, e-objects, $o \in O$, is envisioned to be partitioned into a collection K of *kinds* of objects $k \in K$, each *kind* being a set $k \subseteq O$ of e-objects. Each e-object has to be envisaged as a concrete manifestation with a definite quantity (or even value). The e-objects of some kinds have natural, discrete units, of others are continuous, and of some are 'in between'. We will not address the problem of the choice of a unit, and its predominantly conventional nature here. In the standard literature about measurement[5] this problem is solved by showing that different scales introduced with respect to different units are equivalent in a precise sense. Thus it would be possible to use equivalence classes to get rid of units altogether. However, equivalence classes are mostly handled in terms of their representatives; the use of equivalence classes in the present paper would only introduce additional complications. As mentioned above, our term 'e-objects' comprises not only commodities (inventory, machinery, buildings, etc.) but also such 'social' notions as debts (accounts receivable, payables, bonds, etc.) as well as all aspects of ownership claims (investment in stocks, owner's equity, etc.). The three sets introduced above describe a kind of state $\langle h, o, k \rangle$ with $h \in H$, $o \in O$, and $k \in K$. In state $\langle h, o, k \rangle$, holder h holds (owns or owes or stores or controls, etc.) o numbers or units (or \$-value) of e-objects of kind k – hence o is a quantitative expression in the broadest sense.

D1: S is a *state-space for accounting* if there exist H, O, K such that $S = H \times O \times K$, and

(1) H and O are finite, non-empty sets, and disjoint.

(2) K is a partition[6] of O.

The main data of accounting concerns economic transactions (we denote by ET the set of all *economic transactions*) or *e-transactions*, of which two major kinds ought to be distinguished: (1) *transactions of physical reality* (e.g. transfer of inventory from one place or owner to another – whether for production, distribution or consumption purposes); and (2) *transactions of social reality* (e.g. the creation or termination of a debt claim or of an ownership claim). We may regard an e-transaction as the transfer or conversion of an e-object o of kind k held by holder h at time t, into an object o', of kind k' held by h'. For this we write

$$\langle t, h, o, k, h', o', k' \rangle \in ET .$$

An *exchange* (or exchange transaction) has to be expressed *by two* e-transactions.[7]

An *accounting transaction* or a-transaction, must *not* be confused with an economic or e-transaction. First of all, in contrast to an e-transaction, an a-transaction is *merely a description* and belongs to the *realm of pure concepts*; and secondly, it may describe either a single e-transaction or, more frequently, an exchange or similar combination of two e-transactions.[8] In the case of a *composite accounting transaction* (e.g. the sale of finished goods, perhaps different types of goods, partly against cash, partly against a debt claim), there may be more than one entry on either side (i.e. either on the debt or input side or on the credit or output side or on both).

If we introduce a set T of time instants, an e-transaction

$$\langle t, h, o, k, h', o', k' \rangle$$

may be considered as an element of

$$T \times S \times S$$

where S is a state space ($s \in S$) as defined in D1 above. If the set of *all* e-transactions of a given system is denoted by ET we may write:

$$ET \subseteq T \times S \times S\ .$$

By introducing an ordering relation $<$ for points of time, we offer our definition of an *accounting data system* (ADS) as follows:

D2: x is an *accounting data system* ($x \in ADS$) iff there exist T, $<$, S, ET, and H, O, K such that $x = \langle T, <, S, ET \rangle$ and

(1) $\langle T, < \rangle$ is a finite linear ordering[9]

(2) $S = H \times O \times K$ is a state space for accounting (see *D1*)

(3) $ET \subseteq T \times S \times S$

(4) for all $t \in T$: ET, restricted[10] to t, is a one-one relation on S.

(5) for all $t \in T$ and all $s \in S$: either there is $s' \in S$ such that $\langle t, s, s' \rangle \in ET$ or there is $s' \in S$ such that $\langle t, s', s \rangle \in ET$.

Assumption D2-4 intuitively conveys that the two states involved in an e-transaction correspond with each other in a unique way. If state s (h holds o of k) occurs in an e-transaction, then there is exactly one corresponding state s' which together with s (at time t) forms the e-transaction. In other words, at some instance t it cannot happen that two e-transactions take place involving one state s, on one side, but two different states, on the other side. D2-5 requires that S be chosen minimally with respect to ET. In an e-transaction a state s cannot occur without its corresponding state s'. As indicated above, e-transactions may be modelled or described in an accounting system by means of a-transactions. We will first discuss the most important case of such a system, an accounting system with double-classification.

IV. ACCOUNTS, ACCOUNTABILITY, AND DOUBLE-CLASSIFICATION

The term 'account' has several meanings, but basically it conveys a list of numbers (positive or negative) to be used by a specific (accounting) entity which aggregates these numbers such that at any point of time (i.e. at the end of a stipulated accounting period or arbitrarily chosen sub-period) a unique number, called the *balance* (of the pertinent account, accumulated since the beginning of period p_n) can be determined.[11] A formal explication of this notion will consist in specifying the various items involved (time, list of numbers, balance) and specifying their relations. The 'inner structure' of an account obtained in this way will be expressed in the form of a set-theoretic structure in D4 below.[12] The development of an account is rather trivial: new numbers may accrue to the list at each point of time. The points and periods of time are given by the physical and social reality of e-transactions to be represented through the pertinent accounting system, i.e. by some underlying ADS and the pertinent *account* itself.

The balance or number assigned to this account at a specific time point is usually expressed in a legal tender (e.g. \$, £, DM) and represents the monetary *book value* of this account. Since real systems can distinguish only finite numbers of points and periods of time, we may here safely work with finite sets. An account thus requires first of all, a set of time points $T = \{t_0, t_1, \ldots, t_n\}$ and a set of periods $P = \{p_1, p_2, \ldots, p_n\}$ such that:

(i) t_0 is the beginning of the accounting period p_n and usually of all sub-periods $p_1, p_2, \ldots, p_n$;

(ii) t_1 is the end point of sub-period p_1, t_2 the end point of sub-periods p_2, etc. and t_n the end point of the (total) accounting period p_n.

(iii) p_n is therefore a (total) *accounting period* (usually, but not necessarily, *one year*) beginning at time t_0 (often, but not necessarily, January 1) and ending at time t_n (often December 31). Usually an accounting system continues over *several* accounting periods.

In accounting a 'time point' is usually a day (hence actually a short period), while, a 'period' is usually one or several month(s), or quarters (sub-periods), or a year (customarily addressed as a total period). When *accumulative* sub-periods (first quarter, first two quarters, first three quarters) are compared with each other, it is crucial that the beginning point is always the same (e.g. t_0). The points of time are linearly ordered by a sequence relation

$$t_0 < t_1 < \cdots < t_n .$$

The periods may then be defined as intervals in form of pairs of instants:

$$p_{ij} = \langle t_i, t_j \rangle \quad \text{where} \quad t_i < t_j , \quad 1 \leqslant j \leqslant n , \quad 0 \leqslant i \leqslant n$$

and the sequence relation may be used to define various concepts of comparison for periods $<$. For instance, period p_{ij} is *later than* period p_{rs} iff $t_i < t_r$, and p_{ij} is *longer than* p_{rs} iff the number of points of time between t_r and t_s is smaller than the number between t_i and t_j (this assumes an interpretation of 'points of time' as 'short unit periods' in accordance with accounting practice).

Though a period, in accounting, is at least as important as a point of time, we will (for reasons of conceptual economy) treat only the time point, but *not* the period, as an independent or primitive notion. Since periods can be explicitly defined as indicated, nothing is lost if they are treated as defined notions. Note that under a natural interpretation our points of time refer to extended events, like days.

Next, a mapping c is used to represent the (finite) list of numbers (*entries*) in the account. Formally such a list is a mapping from a finite segment of the natural numbers $N_n = \{1, \ldots, n\}$ into a set of numbers. We take the set R of rational numbers as the range of values for c; $c(i) = \alpha$ is read as 'α is the ith entry in the account'. In order to indicate the point of time (for the *date* at which an entry is made), we use a function δ assigning a point of time to each index i of an entry $\alpha = c(i)$. Formally, δ maps the indices of N_n into T (D4-3 below). The connection between entries and their dates is then given by reading the two equations $\alpha = c(i)$, $\delta(i) = t$ from left to right. Note that at a given point of time several transactions may be performed simultaneously, and consequently several entries may bear the same date in an account.

Finally, and most importantly, each account has a *balance* (which, however, may be zero). The balance of account a at time t may be defined as the sum of all its entries that occurred in the time span from t_0 to t. For reasons to be seen below, we will treat the notion of balance as a primitive (instead of defined) notion, and express its definition as a special axiom. That is to say, we introduce a primitive function B_a, the balance of account a, which maps points of time into numbers (D4-4). By using the 'time scale' δ, we can determine those entries in the list c which have a date smaller than or equal ($\leqslant$) to some given $t \in T$, i.e. the set of entries of $\alpha = \alpha_i = c(i)$ for which $\delta(i) \leqslant t$. By summing up these entries we obtain the common definition for balance at t (D4-6). For the sake of convenience we introduce the following notational conventions:

D3: (*a*) Let $\langle T, < \rangle$ be a finite, linear ordering. By $\leqslant$ we abbreviate the relation on T, defined by $a \leqslant b$ iff ($a < b$ or $a = b$), for all $a, b \in T$. By t_0 we denote the minimal element of T under $<$. If $t \in T$ is different from t_0, we denote by $t - 1$ the maximal $t^* \in T$ such that $t^* < t$.

(b) By N and N_n we denote the set of natural numbers, and natural numbers less or equal than n respectively; and by R and R_0 the set of rational numbers and non-zero rational numbers, respectively.

(c) By π_i we denote the projection of vector $\langle x_i, \ldots, x_n \rangle$ on its ith component, i.e. $\pi_i(\langle x_1, \ldots, x_n \rangle) = x_i$.

D4: a is an *account* iff there exist T, n, R, $<$, c, δ, B_a such that

$$a = \langle T, n, R, <, c, \delta, B_a \rangle \quad \text{and}$$

(1) $n \in N$ and $\langle T, < \rangle$ is a finite linear ordering

(2) $c: N_n \to R_0$

(3) $\delta : N_n \to T$

(4) $B_a: T \to R$

(5) for all $i, j \in N_n: i \leqslant j$ iff $\delta(i) \leqslant \delta(j)$

(6) for all $t \in T: B_a(t) = \sum_{\delta(i) \leqslant t} c(i)$

D4-5 states that the two orderings, $\leqslant$ in the natural numbers, and $\leqslant$ among time points, fit together under δ. The definition of B_a in D4-6 is just one among many alternatives.[13]

A single account is a rather simple device for merely describing quantitatively and/or in monetary terms a series of economic events and their resulting monetary balance. Only the combination of several accounts to a system gives rise to a more interesting structure, particularly under the aspect of double-entry or, more precisely, *double classification*. Let us consider a simple, real-life situation: a firm (entity e), in recording its transactions, uses the set of accounts $A_e = \{a_1, a_2, a_3\}$, $a_i = \langle T_i, n_i, R, <_i, c_i, \delta_i, B(a_i, \cdot) \rangle$, consisting of three accounts: Inventory (a_1), Cash (a_2), and Owner's Equity (a_3). Suppose at time t its economic state consists merely of an inventory worth 1,000 monetary units (we shall use the \$ as monetary unit or omit it where any misunderstanding is excluded), but no cash. The totality of its accounts' balances (i.e. what accountants call the 'trial balance' $TB(t) = \langle B(a_1, t), \ldots, B(a_3, t) \rangle$ at t will then be: $B(a_1, t) = +1{,}000$, $B(a_2, t) = 0$, $B(a_3, t) = -1{,}000$. In terms of c we have $c_1(1) = 1{,}000$, $c_3(1) = -1{,}000$ while $c_2(1)$ is as yet undefined (0 is not entered, except as a balance). The positive figure $+1{,}000$ of inventory indicates in which asset the capital is invested (inventory is here the only capital good), the negative figure $-1{,}000$ indicates the source of this capital (here the only source is owner's equity, i.e. so far, capital derives only from the owner, but not yet from creditors).

To render plausible the relationship between the purely conventional use of the 'negative' (Cr: credit entry) and the 'positive' (Dr: debit entry), one may imagine the investment activity of the owner: e.g. he invests \$1,000 (*output* from him, hence negative) into his firm as inventory (where it is an *input*, hence positive).

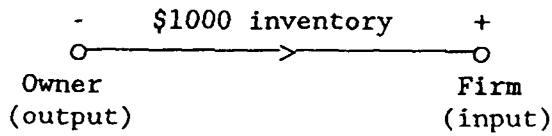


Fig. 1.

Thus the owner's equity account held in the set of accounts A_e of firm or entity e reflects the output from the owner (which is his *social* contribution or ownership claim) while the inventory account reflects the *physical* input of some asset (e.g. inventory) by the owner.

If the firm sells its total inventory for \$1,500 at time $t+1$, its accountant will proceed as follows: First, enter \$1,500 positively in cash account a_2, hence $c_2(1) = 1{,}500$ and $B(a_2, t+1) = B(a_2, t) + 1{,}500 = (0 + 1500)$; second, there is an output (hence negative) of inventory from the firm at cost value \$1,000, hence $c_1(2) = -1{,}000$ and $B(a_1, t+1) = B(a_1, t) - 1{,}000 = (1{,}000 - 1{,}000) = 0$, but simultaneously one has to recognize that, due to this profitable sale (we assume that no other costs are involved), the ownership claim on the entity has increased by \$500, and since the owner's claim is recorded negatively (i.e. as a credit entry), $c_3(2) = -500$ and the balance is $B(a_3, t+1) = B(a_3, t) - 500 = (-1{,}000 - 500) = -1{,}500$. Of course, alternative procedures can be imagined. Justification for the one just described can be given in terms of the notion of *accountability*.[14] Any set of debit values is always *accountable* in terms of its corresponding set of credit values and *vice versa*. But at this stage we do not so much focus on the normative background as on the underlying logical structure.

As we have seen, the point of double-classification is that each a-transaction is represented by two entries of opposite sign at least in two different accounts *of the same firm*. With few exceptions (e.g. in constructing consolidated financial statements) accountants concentrate on a single entity at a time. However, e-transactions often involve different entities. In order to obtain an adequate representation of accounting data systems we have to consider systems of accounts distributed over different entities. We introduce the notion of an account *belonging to* entity e which we abbreviate by $b(a, e)$. If A and E denote a set of accounts and a set of entities respectively, we may take b as a binary relation $b \subseteq A \times E$. In analogy to D2 we introduce a state-space $C(e)$ for a set of accounts of an entity e (D5-c below), and similarly for a set of accounts A, used by all members of a set of entities E ($e \in E$) – see D5-b. Intuitively a state $\langle a, i, \alpha \rangle$ in $C(e)$ as well as a state $\langle e, a, i, \alpha \rangle$ in $C(E, A)$ conveys the information that in account a, belonging to entity e, the ith entry is α (or, put differently, if $a = \langle T, n, \ldots, c, \ldots \rangle$ then $i \leq n$ and $c(i) = \alpha$).

D5: (*a*) If $a = \langle T, n, \ldots, B_a \rangle$ is an account, we write $T^a, n^a, \ldots, \delta^a$, B_a to denote the components of a. Instead of $B_a(t)$ we also write $B(a, t)$.

(b) If A is a set of accounts, E a set of entities, and $b \subseteq A \times E$, then the *state-space* of A with *respect to* E (and b), $C(E, A)$, is defined *by* $C(E, A) = \{\langle e, a, i, \alpha \rangle | e \in E \wedge a \in A \wedge b(a, e) \wedge i \leq n^a \wedge \alpha = c^a(i)\}$.

(c) Let A, E, and b be as in (b) and $e \in E$. The *state space* of e (in A with respect to b), $C(e)$, is defined by $C(e) = \{\langle a, i, \alpha \rangle \mid a \in A \wedge b(a, e) \wedge i \leq n^a \wedge \alpha = c^a(i)\}$. If $d\colon C(E, A) \rightarrow C(E, A)$, then d_e: $C(e) \rightarrow C(e)$ is defined by $d_e(a, i, \alpha) = d(e, a, i, \alpha)$.

To express double classification, we use a function d mapping states into 'corresponding' states such that the d-image of state s is that state which constitutes the *counter-entry* (bearing the same amount with the opposite sign). We require that in the domain of accounts of each entity e the states are mapped onto each other bijectively D6-7. This means that to each 'entry' (state) in some account of e there corresponds exactly one counter-entry in some other account of e. In D6-8 below, we state the requirements typical for double classification. The two accounts involved have to be different (D6-8.1), the points of time associated with the entries α, α' are the same (D6-8.2), and the numerical entries on both sides are the same apart from the opposite sign (D6-8.3).

D6: x is a *double classification accounting system* ($x \in AS2$) iff there exist A, E, T, $<$, b, d such that $x = \langle A, E, T, <, b, d \rangle$ and

(1) A is a finite set of accounts.

(2) E is a finite, non-empty set.

(3) $\langle T, < \rangle$ is a finite, linear ordering.

(4) $b \subseteq A \times E$.

(5) $d: C(E, A) \rightarrow C(E, A)$ – see D5-b.

(6) for all $a \in A: \langle T^a, <^a \rangle = \langle T, < \rangle$

(7) for all $e \in E$, the mapping $d_e: C(e) \rightarrow C(e)$ is bijective – see D5c

(8) for all $e \in E$ and all a, a', i, j, α, α' iff $d_e\ (a, i, \alpha) = \langle a', j, \alpha' \rangle$ then

(8.1) $a \neq a'$, (8.2) $\delta^a(i) = \delta^{a'}(j)$, and (8.3) $\alpha = -\alpha'$.

The reader acquainted with accounting may miss the central requirement of 'accounting equilibrium' in D6, namely the stipulation that in each entity e, the sum total of all balances is zero: (1) $\Sigma_{b(a,e)} B(a, t) = 0$. But as already mentioned, the point of double classification is that this equilibrium is a consequnce of D6. This is expressed in:

THEOREM 1. *Let* $x = \{A, \ldots, d\} \in AS2$, $e \in E$ *and* $t \in T$, *then* $\Sigma_{b(a,e)}(a, t) = 0$.
B

The proofs of this and the following theorems are given in the Appendix. The *converse*, namely that $\Sigma B(a, t) = 0$ implies D6-7 and D6-8, does *not* hold, which may be seen from a counter example. It ought to be stressed that D6 represents only a special class of accounting systems, namely those with double-classification. In the general case (to be treated in Section VI), the content of T1 would have to be used as a central axiom. In the present case there is a strong connection between the balances B_a occurring in different accounts and the 'correspondence' function d_e of states ('entries'). For each entry in a, the function d_e precisely locates the counter entry in the corresponding account, and by summing up all those counter entries we obtain, of course, the same balance but with reverse sign. This we may express in

THEOREM 2. *Let* $x = \langle A, \ldots, d\rangle \in AS2$, $e \in E$, $t \in T$ *such that* $b(a, e)$. *Then*

$$B(a, t) = -\sum_{\delta^a(i) \leqslant^a t} \pi_3(d_e(a, i, c^a(i)))$$

Another theorem makes more explicit the *accountability principle* which stipulates that all inputs have to be accounted for in terms of all output (or *vice versa*). One of us has previously shown that this principle has close affinity with the conservation principles of physics (which may also be regarded as an accountability principle).[15]

THEOREM 3. Formal Accountability Principle, Version (a): *Let* $x = \langle A, \ldots, d\rangle \in AS2$, *and* $e \in E$. *For* $t \in T$ *let* $SE(e, t)$ *be the sum of all entries*, $c^a(i)$ *such that* $b(a, e) \wedge \delta^a(i) = t$. *Then for all* $t, t' \in T$ *and all* $e \in E$: $SE(e, t) = SE(e, t')$

Formal Accountability Principle, Version (b): *Let* x, e *and* t *be as in* (*a*). *Define* $SE(e, t, +)$ *and* $SE(e, t, -)$ *as the sum of all* $c^a(i)$ *such that* $b(a, e) \wedge \delta^a(i) = t \wedge c^a(i) > 0$, *and the sum of all* $c^a(i)$ *such that* $b(a, e) \wedge \delta^a(i) = t \wedge c^a(i) < 0$, *respectively, then* $SE(\cancel{s}\,e, t, +) =$ $-\,SE(e, t, -)$.

V. REPRESENTATION AND THE PROBLEM OF VALUE

Systems of accounts serve to maintain a chronological record of *past* (and in "budgeting" even expected) economic events, and represent the pertinent data that constitute an economic transaction system. We are now in a position to describe in detail how this works. The e-objects are represented in the accounts by their monetary value; the way the e-objects are distributed within a specific entity is represented by a corresponding classification of accounts. In other words, to a given ADS, $x = \langle T, <, H \times O \times K, ET\rangle$, we assign:

(i) a set A of accounts such that to each kind of e-objects $k \in K$ there corresponds at least one account $a \in A$;

(ii) a set E of entities, let us assume, one for each holder $h \in H$, and

(iii) to any two different e-objects o, o' occurring in an e-transaction of ET, two entries in A.

More precisely, we define an accounting morphism θ from some given $x \in ADS$ to some given $y \in AS2$ as follows:

D7: (*a*) Let $x = \langle T, <, H \times O \times K, ET \rangle \in ADS$ and $y = \langle A, E, T', <', b, d \rangle \in AS2$ such that $T = T'$ and $< = <'$. θ is an *accounting morphism* from x to y iff there exist φ, ψ, and ν such that:

(1) $\theta : ET \to (T \times C(E, A) \times C(E, A))$

(2) $\varphi : H \to E$ is bijective

(3) $\psi : A \to K$

(4) $v : O \to R_0$

(5) for all $t \in T$: θ, restricted to t, is one–one

(6) for all t, t', h, h', o, o', e, e', a, a', i, j, α, β: if $\theta(t, h, o, k, h', o', k') = \langle t', e, a, i, \alpha, e', a', j, \beta \rangle$, then:

(6.1) $t = t'$

(6.2) $\varphi(h) = e$ and $\varphi(h') = e'$

(6.3) $\psi(a) = k$ and $\psi(a') = k'$

(6.4) $v(o) = \alpha$ and $v(o') = \beta$

(6.5) $b(a, e)$ and $b(a', e')$

(6.6) $i \leqslant n^a$, $j \leqslant n^{a'}$ and $\delta^a(i) = \delta^{a'}(j) = t$

(6.7) $c^a(i) = \alpha$ and $c^{a'}(j) = \beta$

(b) We say that $y \in AS2$ represents $x \in ADS$ iff there is some accounting morphism θ from x to y.

In part (a) of this definition the function ψ assigns to each account a specific kind of economic object or objects, namely the kind for which this account is held in the pertinent entity. Usually ψ will be a many-one function, since different accounts will be held by different entities for the same kind of goods. We may think of E as the collection $\{\varphi(h) \mid h \in H\}$ of entities e held by some holder in H. The accounting morphism θ is required to operate one–one at each point of time. To each instant t and to any two states $\langle h, o, k\rangle$, $\langle h', o', k'\rangle$ forming an e-transaction, θ assigns a triple consisting of the same instant (D7a–6.1) and two states $s = \langle e, a, i, \alpha\rangle$ and $s' = \langle e', a', j, \beta\rangle$. By D7–6.2 the corresponding holders and entities h, h' and e, e' are mapped onto each other by means of φ; and by D7a–6.3 the accounts a, a' are mapped onto their corresponding kinds of goods k, k' by ψ respectively. Function v now assigns explicitly to each object or quantity in x a number $v(o)$, the value of o. By D7a–6.4 the numbers α, β occurring on the right-hand side of θ are required to be just the values of o and o' respectively. D7a–6.5 to D7a–6.7 say that s, s', are states as defined in D5. Finally, D7a–6.6 expresses the identity of the instants pertaining to the two entries α and β in a and a' respectively. Note that we assume the time-orderings in x and y to be the same, which (by D6–6) implies that all accounts of y also have the same time-ordering.

The question whether any system of e-transactions (i.e. any accounting *data* system ADS) can be represented through a double classificational accounting system ($AS2$) is answered in the affirmative by the following theorem.

THEOREM 4. *For each $x \in ADS$ there exists some $y \in AS2$ such that y represents x.*

The second, more difficult question, is: how many different double classificational accounting systems can be constructed representing one given $x \in ADS$, how can these alternative systems be transformed into each other, and how can they be compared with each other regarding cost-efficiency? It is one of the major empirical problems of accounting to determine *relevant* values, i.e., the numbers $v(o)$, in a *cost-efficient* way for representing economic transactions and their results. Innumerable decisions (from resource allocation, ownership and income de-termination, debt control, the monitoring of stewardship, liquidity and bankruptcy issues, etc. to taxation) depend on those values. But as the items A, E, T, $<$, b, d are all determined more or less directly by the *ADS*, the only items *not* yet determined are the numerical (or \$) values to be written into the accounts and financial statements. And this we address as *the valuation problem* to be referred to shortly in Section VII. Since the internal structure of accounting data systems and the requirements on representation do not narrow down the range of values in any interesting way, we can prove the following theorem:

THEOREM 5. *Let $x \in ADS$, $y \in AS2$ be such that y represents x. Change the v-values used in the accounting morphism to y as follows: to each $o \in O$ such that, for some t, h, k, h', k', o', $\langle t, h, o, k, h', o', k' \rangle \in ET$, replace the value of $v(o)$ by some arbitrary positive number $v^*(o)$ and $-v(o')$ by $-v^*(o')$. Then there is some y' and some representation extending v^* which is an accounting morphism of x into y'.*

In other words, the only restriction imposed on the v^* values by our axioms is about 'corresponding pairs': if one v-value is chosen, the 'corresponding' value has to be fixed accordingly. But the choice of one of the two is not restricted at all.

We can easily transform the prevous definitions into the format of what is called a *theory-element* in structuralist meta-theory.[16] We may define an *accounting model* to be a 'mixed' structure $\langle T, <, H, O, K, ET, A, E, b, d, \theta \rangle$ such that $x = \langle T, <, H \times O \times K, ET \rangle \in ADS$, $y = \langle \cancel{A, T, <, E;}\ d \rangle \in AS2$ and θ is an accounting morphism from x into y. If this use of the term 'model' is acceptable (despite the preference of one of us for restricting this term to conceptual, i.e. 'representational', items only) then a *potential model* is a structure of the same type but satisfying only D2, D4–1 to D4–4, D6–1 to D6–4, and D7a–1 to D7a–4. The 'representing' components A, E, d, θ in such a model may then be regarded as theoretical terms, and by omitting them from potential models one obtains *partial potential models* that correspond to our *ADS*s. If we assume that certain concrete *ADS*s are delineated to form the basis for *intended applications* of the present formalism, then we have specified all items making up a theory-element (neglecting various constraints and links).

$\otimes$ $\langle A, E, T, <, b, d \rangle$

According to the structuralist account, the *empirical claim* associated with a theory-element is such that each intended application can be augmented by theoretical terms so that the resulting structure is a model. In our case this would be the claim that each intended *ADS* i.e. $x = \langle T, <, H \times O \times K, ET \rangle$ can be made to correspond to a model by means of suitable A, E, b, d, θ. That is, to each intended *ADS* we can find a representing system of accounts.

Summarizing this discussion we may say that the basis of accounting is representable in the form of an empirical theory; but because of its teleological nature (i.e. its purpose orientation) it has not so much a positive but rather a *normative-empirical* content. This feature (as well as its historical origin) is shared with some mathematical and other conditionally normative theories (e.g. in operations research and economics). Roughly, the difference between theories of the latter type and theories in the natural sciences is this: in theories like accounting, the *applicability* of the non-theoretical conceptual apparatus to a real system or situation already entails that the system comes out as a proper model of the theory. In other words, the theoretical models add very little to what is imposed on a given situation by applying the non-theoretical concepts to it. This is the reason why 'basic accounting' and other teleological theories, seem to hold a position in between mathematical theories and positive empirical theories.

To shortly illustrate further aspects, we refer to the example of Sec. IV of a firm using the accounts a_1 (Inventory), a_2 (Cash) and a_3 (Owners' Equity) with balances $B(a_1, t+1) = 0$, $B(a_2, t+1) = 1{,}500$, $B(a_3, t+1) = -1{,}500$. Suppose at time $t+2$ the firm might borrow cash in the amount of \$2,000 and at $t+3$ buy inventory for \$1,300. In this case we have to open first a new account, called 'creditors' or 'payables' a_4 and credit the borrowed amount to this account, which then will show a balance $B(a_4, t+2) = -2{,}000$ – its balance is 'negative' since payables is a credit account like owner's equity from which an output is derived for an input to the firm:

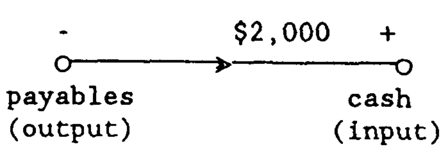


Fig. 2.

The counter entry will be an equivalent positive (Dr) amount to the cash account, the balance of which will be $B(a_2, t+2) = B(a_2, t+1) + 2{,}000 = 1{,}500 + 2{,}000 = 3{,}500$. Then we have to record the purchase of inventory, hence a positive (Dr) entry of \$1,300 to the inventory account, the balance of which will be $B(a_1, t+3) = B(a_1, t+2) + 1{,}300 = B(a_1, t+1) + 1{,}300 = 0 + 1{,}300 = +1{,}300$, while its counter entry is in the cash account, the balance of which will be $B(a_2, t+2) - 1{,}300 = 3{,}500 - 1{,}300 = 2{,}200$. Since the transaction of borrowing (e.g. \$2,000) is here quite separate from that of buying inventory, several time points and accounts are involved. Even more complex situations arise in cases where the cost of fixed assets, like machinery (e.g. account a_5), have to be depreciated, i.e. allocated over *several* accounting periods (including present and future periods).

VI. GENERALIZED SYSTEMS OF ACCOUNTS

Double classification accounting systems form only a special case of accounting systems – though by far the most important one. In order to achieve a completely general characterization we may essentially use the formalism of D6 but generalize the axioms.[17] Accordingly, we generalize the function d used in D6, and replace d by a relation g relating whole sets of states with each other. The particular description of two such sets of states (sets of 'entries') is obtained from, and corresponds to, a particular procedure in the sense used before. We therefore call g a *generalized procedure*. Formally, g will be a relation between sets of states, i.e. a relation on the power set Po of $C(E, A)$:

$$g \subseteq Po(C(E, A)) \times Po(C(E, A))$$

The points of time associated with different entries occurring in g now may be different, and the requirement of one–one correspondence between two single entries under d within one firm (D6–7) has to be dropped. Instead, we require that all states occurring in sets related by g refer to the same entity (D8–7 below). Furthermore, the characterization of double classification (D6–8), which is just one special procedure, no longer needs to be maintained in a more general presentation. Instead, we require the general principle of balance or equilibrium within each entity e (see D8–8 below). We are then led to the following definition:

D8: x is an *accounting system* (in general) iff there exist A, E, T, $<$, b, g such that $x = \langle A, E, T, <, b, g\rangle$ and

(1) A is a finite set of accounts

(2) E is a non-empty, finite set

(3) $\langle T, <\rangle$ is a finite linear ordering

(4) $b \subseteq A \times E$

(5) $g \subseteq Po(C(E, A)) \times Po(C(E,A))$

(6) for all $a \in A$: $\langle T^a, <^a\rangle = \langle T, <\rangle$

(7) for all X, Y, $z_1, \ldots, z_4$, $e_1, \ldots, e_4$: if $g(X, Y)$, $z_1, z_2 \in X$, $z_3, z_4 \in Y$ and $z_i = \langle e_i, \ldots\rangle$ for $i \leq 4$ then $e_i = e_j$, for all i, $j \leq 4$

(8) for all $e \in E$ and $t \in T$ $\sum_{b(a,e)} B(a, t) = 0$.

The systems of data represented by such systems of accounts are the same as before, and so is the concept of an accounting morphism.

The situation with respect to empiricity is unchanged. We can formulate analoga to Theorems 4 and 5 stating that each *ADS* can be represented by some system of accounts, and that within each cluster of g-related values, one value may be changed arbitrarily and the others adjusted so that the result still is a system of accounts representing the same system of data.

By putting together the previous definitions we obtain the core model for accounting. We speak of a core model because it does not yet contain accounting in all its specializations – specializations correspond to accounting interpretations; for the latter see Mattessich (1972). The model provides a core from which further specifications of accounting can be obtained. The picture we have here in mind is that of a theory-net consisting of a core element and various specializations of the core. Recent metatheoretical studies suggest that the form of such nets (in contrast to an 'unstructured' set of axioms) is appropriate to represent empirical theories.[18]

D9: x is a *core model of accounting* iff there exist T, $<$, H, O, K, ET, A, E, b, g and θ such that $x = \langle T, <, H, O, K, ET, A, E, b, g, \theta\rangle$ and

(1) $\langle T, <, H \times O \times K, ET \rangle$ is an ADS

(2) $\langle A, E, T, <, b, g \rangle$ is an accounting system

(3) θ is an accounting morphism from $\langle T, <, H \times O \times K, ET \rangle$ to $\langle A, E, T, \underset{<}{\cancel{<}}, b, g \rangle$.

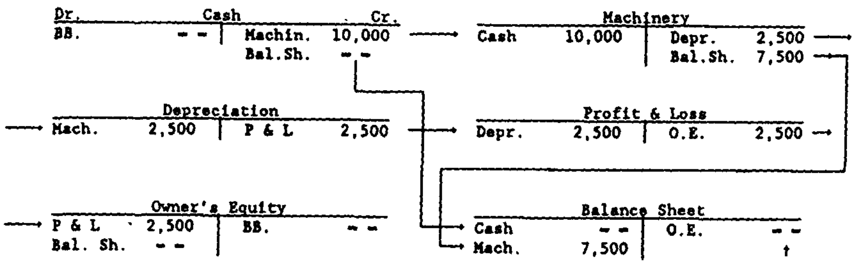
VII. SPECIALIZATIONS

We close this paper by considering some specializations of the core model described in Section VI. For reasons of space the different specializations can only be outlined in a rather sketchy way. A full treatment has to be reserved for future work. We claim that all special methods and procedures used by accountants can be obtained from the core model by appropriate specializations. By putting together our core model and all those specializations one would obtain a 'theory-net' of nearly the same type as those extracted from theories of other sciences.

The specializations proceed mostly along one of three lines. One type of specialization works by putting further requirements on the procedure-relation g in such a way that special procedures of accounting can be characterized in detail. The second kind locks in at the accounting morphism θ, and the value function v. These specializations mainly consist of methods of how to determine or adjust values (e.g. of durable equipment over time). A third line of specialization consists in requiring a certain minimal (or other more complicated) structure of special accounts. Let us look at these three branches in turn.

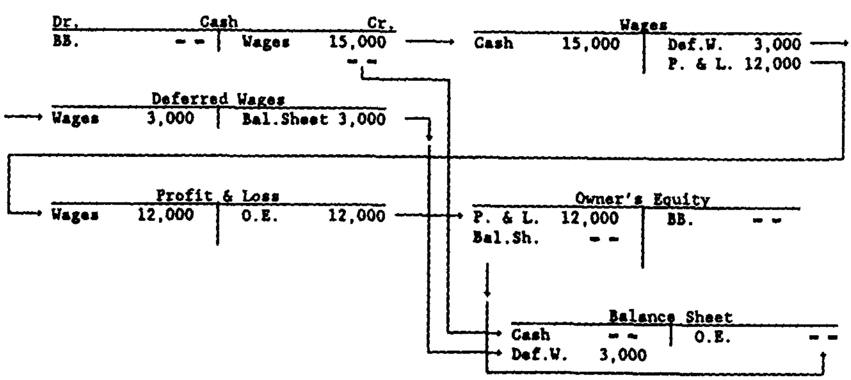
We start by indicating how D8 has to be specialized in order to obtain double classification. This is done in three steps. First, the procedure g is required to operate on singletons: $g(\{\langle e, a, i, \propto \rangle\} = \{\langle e', a', j, \beta \rangle\})$ which of course correspond precisely to the states used in D6. Second, g restricted to each e is required to be bijective on the set of such singletons, and third, double classification is expressed as in D6-8, i.e., if $g(\{\langle e, a, i, \propto \rangle\}) = \{\langle e, a', j, \beta \rangle\}$, then $a \neq a'$, $\delta^{a}(i) = \delta^{a'}(j)$ and $\propto \, = -\beta$. It may be noted that this kind of 'specialization' is somewhat more general than the standard notion.[19]

A second specialization is obtained by describing the procedure of depreciation accounting. As an example, consider the purchase (cash) of machinery for \$10,000 with depreciation of machinery (linearly) over 4 years ($2,500 each year). The corresponding procedure in accounting terminology may be represented like this:



By collecting the debit entries in these six accounts to yield a set of states X and the credit entries to yield another set of states Y we may subsume the overall procedure under our core model by writing $g(X, Y)$. The relation g now might be precisely described just by going through the above schemes and abstracting from the particular numbers involved (4, 10,000, 2,500, 7,500). Of course, the procedure might be extended to future periods in which case the sets X and Y, corresponding to the entries involved, become larger and spread over time. Realistically, this is involved in depreciation whereby the accountant may obtain information that depreciation is continued in the following period at the same rate.

Another special procedure is given by accrual (and deferral) accounting. Consider the payment of wages (Dec. 27–Jan. 2) of $15,000 such that $3,000 (for Jan. 1 and 2) are deferred. The accountant's scheme looks like this:



Again, the debit entries involved yield a set X and the credit entries a set Y of states such that $g(X, Y)$, and a precise generalized description of the procedure could be extracted from the example.

The second line of specializations is given by various valuation methods. Here, we have first the *historical (acquisition) cost method.* The original amount spent for an item is used as its value (it may be adjusted for depreciation at the end of the period, provided it is a depreciable item like Machinery, Building, or even Accounts Receivable). Land or Inventory usually do not fall under regularly depreciable items.

Second, there is the *current (entry) or replacement value method* which adjusts (by specific price indices or more directly) the acquisition cost such that at the end of a period the balance corresponds to the replacement value (i.e. to the amount which would have to be paid if that asset, or group of assets, in its present state were presently acquired by the firm). Similar to this is the third or *current exit value method* which adjusts the acquisition cost such that at the end of a period the balance corresponds to the net amount for which the firm could sell this asset.

A fourth method of valuation is the *general price-level adjustment or constant dollar method* which adjusts the balances of all 'non-financial' accounts (i.e. excluding cash, receivables, payables, mortgages and other obligations) with a general inflation/deflation index – usually at the end of a period. This method is occasionally combined with the second method.

Fifth, we mention the *present value method* which adjusts the acquisition cost such that the ending balance corresponds to the subjectively evaluated discounted future net-revenues of the respective asset. This 'economic method' has mainly significance in finance and managerial accounting, but has only limited use in the practice of public financial statement presentation. Sometimes the current value method is considered as an approximation to it.

These five valuation methods can be made precise by reference to the value function v which is part of the accounting morphism θ. Essentially, they bring in new features not explicitly definable in the core model, namely the different ways in which values are determined and adjusted. This, however, is also the case in other theory-nets, and is compatible with the notion of specialization appropriate for such nets (in mechanics, for instance, Hooke's Law refers to the spring constant which is not a primitive in the core model given by Newton's laws). Of course, these five methods do not exhaust all possibilities, and sometimes combinations of them and other empirical aspects are encountered. Of particular importance are the various types of gains which may arise (e.g. operating gains versus holding gains, monetary holding gains versus non-monetary gains, realized versus unrealized holding gains).

Finally, a third kind of specialization requires the specification of the types of accounts and statements needed and to be used in a particular accounting system (these are usually found in the charts and master charts of accounts, among which Schmalenbach's 'Kontenrahmen' has attained particular prominence) – see Mattessich (1964/77), pp. 66–68, 517.

All those methods, procedures, and hypotheses arise in the domain of specializations of the core model; for this reason the core model itself is a purely analytical, and in a way, 'trivial' model. But this ('trivial' core model versus 'interesting' special applications) is typical of mature and developed theories in general. And the aim of this paper was to bring out the details and the structure of the core model; for it is the core model that provides the unity for the net of specializations, and thus constitutes the prerequisite for any further work.

APPENDIX

Proof of Theorem 1

Let x, e, t be given. Let $A^* = \{a_1, \ldots, a_m\}$ be the set of those accounts for which $b(a, e)$ holds, and $J = \{j_i, \ldots j_k\}$ be the set of all pairs $\langle a, i\rangle$ for which $\delta^a(i) \leqslant t$, $a \in A^*$. For any $\langle a, i\rangle \in J$, there is, by D6–7, exactly one $\langle a^*, j\rangle \in J$ such that $d_e(a, i, c^a(i)) = \langle a^*, j, c^{a^*}(j)\rangle$ and $c^a(i) = -c^{a^*}(j)$.
This defines a function $f: J \to J$ such that

(1) $\quad c^{a^*}(j) = -c^a(i)$ whenever $f(a, i) = \langle a^*, j\rangle$.

Clearly, f is bijective. Let $J^* \subseteq J$ be such that ($c^a(i) > 0$ iff $\langle a, i\rangle \in J^*$). Now consider

$$\sum_{b(a,e)} B(a,t) = \sum_{b(a,e)} \sum_{\delta^a(i) \preccurlyeq t} c^a(i)$$

$$= \sum_{\substack{j \in J \\ j=\langle a,i\rangle}} c^a(i) = \sum_{j \in J^*} c^a(i)m + \sum_{\substack{j \in J \setminus J^* \\ j=\langle a,i\rangle}} c^a(i)\,.$$

To each summation index $j = \langle a, i\rangle$ in the first sum there is exactly one index $j^* = \langle a^*, i^*\rangle$ such that $f(a, i) = \langle a^*, i^*\rangle$, and by this correspondence all indices of the second sum are used up. We therefore may write the second sum in the form $\Sigma_{\substack{\langle a,i\rangle \in J^* \\ \text{and } f(a,i) \\ =\langle a^*,i^*\rangle}} c^{a^*}(i^*)$ which, by (1) is equal to $\Sigma_{\langle a,i\rangle \in J^*} - c^a(i)$.

So $\Sigma_{b(a,e)} B(a,t) = \Sigma_{\langle a,i\rangle \in J^*} c^a(i) + \Sigma_{\langle a,i\rangle \in J^*} - c^a(i) = 0\,.$ ■

This may seem to be a complicated way of proving such a simple, pretty selfevident, theorem; but this proof shows the minute structural relations, not unlike the details revealed by a microscope.

Proof of Theorem 2

For $\langle a, i, c^a(i)\rangle$ with $\delta^a(i) \preccurlyeq t$ there exists exactly one $\langle a^*, j, \alpha\rangle$ such that $d_e(a, i, c^a(i)) = \langle a^*, j, \alpha\rangle$, and by D6–8 $\alpha = \pi_3(d_e(a, i, c^a(i))) = -c^a(i)$. So $B(a,t) = \Sigma_{\delta^a(i) \preccurlyeq t} c^a(i) = -\Sigma_{\delta^a(i) \preccurlyeq t} \pi_3(d_e(a, i, c^a(i)))$. ■

Proof of Theorem 3

(a) Let J be the set of all pairs $\langle a, i\rangle$ with $b(a, e)$ and $\delta^a(i) = t$, $J^* = \{ j \mid j = \langle a, i\rangle \in J \wedge c^a(i) > 0\}$. Then, as in the proof of Theorem 1, $\Sigma_{b(a,e) \wedge \delta^a(i)=t} c^a(i) = \Sigma_{\langle a,i\rangle \in J} c^a(i) = \sum_{\langle a,i\rangle \in J^*} c^a(i) + \sum_{\langle a,i\rangle \in J^*} - c^a(i) = 0$ by D6–8.2. This holds for any $t \in T$ ■

(b) Is proved by the same method of splitting up the set of indices.

Proof of Theorem 4

Let $x = \langle T, <, H \times O \times K, \quad ET\rangle \in ADS$. We define $y = \langle A, E, T', <, b, d\rangle$ as follows: $T' = T$, $<' = <$, $E = H$. For each $e \in E$, $k \in K$ we introduce two accounts $a(e, k, +)$ and $a(e, k, -)$, and we define $b(a(e, k, \cdot), e)$ for all those. Since O and H (and therefore K) are finite, the set A of accounts thus introduced is also finite. Starting from some arbitrary enumeration of the tuples $\langle t, h, o, k, h', o', k'\rangle \in ET$ we define the entries of the accounts as follows. For each tuple $\langle t, \ldots, k'\rangle \in ET$ we choose two positive numbers $v(o)$ and $v(o')$, and we set $c^{a(e,k,+)}(i) = v(o)$, $c^{a(e,k,-)}(i) = -v(o)$, $c^{a(e',k',+)}(i) = v(o')$ and $c^{a(e',k',-)}(i) = -v(o')$ where i in each case is the smallest natural number that has not yet been used as an index in the corresponding account, and $\delta^x(i) = t$ for $x = a(e, k, +), \ldots, a(e', k', -)$. We define $d: C(E, A) \rightarrow C(E, A)$ by $d(e, x, i, \alpha) = \langle e, y, j, -\alpha\rangle$ where y is $a(e, k, -)$ if $x = a(e, k, +)$ and $y = a(e, k, +)$ if $x = a(e, k, -)$ and i, j are the corresponding minimal indices. By construction $d_e: C(e) \rightarrow C(e)$ is bijective for each $e \in E$, and D6–8 is satisfied. So $y = \langle A, E, T, <, b, d\rangle \in AS2$.

For each $x = \langle t, e, o, k, e', o', k'\rangle \epsilon \mathrm{ET}$ we define $\theta(x) = \langle t, e, a(e, k, +), i, v(o), e', a(e', k', +), j, v(o')\rangle$ where i, j are chosen as in the construction above. Further, we define $\psi: A \rightarrow K$ by $\psi(a) = k$ if a has the form $a = a(e, k, \cdot)$, and $\psi: H \rightarrow E$ to be identity. To e-objects o not occurring in e-transactions we assign arbitrary numbers $v(o)$. By construction θ, restricted to t, is one–one, and D7a–6 is satisfied. ■

Proof of Theorem 5

Let $x \in ADS$ and $y \in AS2$ be given such that y represents x, and let y^* be the modification of y as described in the Theorem. Clearly, D7 is not affected by this modification. We have to show that y^* still satisfies D6. This is seen by replacing α and α' in D6–8 by $v^*(o)$ and $v^*(o')$, respectively. Under the assumption of the Theorem, $v^*(o) = -v^*(o')$, so D6–8.3 is satisfied also after the replacement. ■

NOTES

[1] W. Balzer is Professor at the Seminar for Philosophy, Logic and Epistemology of the University of Munich, and R. Mattessich is Arthur Andersen & Co. Professor emeritus of the Faculty of Commerce and Business Administration of the University of British Columbia, Vancouver (Canada). The latter wishes to acknowledge support for his contribution by the Social Sciences and Humanities Research Council of Canada. We are also grateful to Dr. Alfred Wagenhofer, of the University of Technology of Vienna, for reading a draft of this manuscript and making some valuable suggestions.

[2] The reason for this pessimism does not lie in the limitations of the axiomatic method, but in those of practicing accountants. Axiomatic systems of accounting might prove highly useful for the creation of the *Conceptual Framework* – cf. Moonitz (1961/82) and Sprouse and Moonitz (1962/82) – on which the Financial Accounting Standards Board (FASB) of the USA has been working for over a decade. But sophisticated scientific tools have rarely, if ever, been taken into consideration by the FASB or other professional accounting bodies.

[3] For a German version, see Mattessich (1970), and a Japanese translation was published in 1972 and 1974.

[4] For details see Mattessich (1987b, 1989a,b) and Schmandt-Besserat (1982, 1988).

[5] Some e-objects, such as wheat, petrol, etc., are bulk goods, and their quantities are measured on a continuous scale (at least theoretically), while the quantities of individual goods, like apples or cars, are measured in discrete units. But the values of all of them are measured in terms of a legal tender or similar 'monetary' unit. E.g. Krantz *et al.* (1971).

[6] A *partition* of O is a collection of non-empty, disjoint subsets of O which, together, exhaust all of O.

[7] Thus an e-transaction is something *more basic* than a barter: in the latter, an e-object o of kind k is handed over, in *quid pro quo*, from one holder to another (hence $h \neq h'$, but $o = o'$ and $k = k'$), but in *a sale* (exchange against money or a debt claim) a commodity is transferred from holder h to h' (hence $h \neq h'$) and e-object o (e.g. a commodity) of kind k is exchanged against e-object o' of kind k' (a debt claim). Whether $k = k'$ or $k \neq k'$ cannot be said at this stage; if o expresses the sales value of the seller, and o' the *cost value* (not the resale value) of the buyer it is likely that $o = o'$.

[8] For example, the double entry:

		Dr:	Cr:
(1)	Work in Process – Material	5,000	
	Inventory – Material		5,000

is an a-transaction *describing* a single *physical e-transaction*, while the following entry is an *a-transaction* that does *not* correspond to a single e-transaction:

(2)	Inventory – Finished Goods	8,000	
	Accounts Payable		8,000

The latter entry is the *combination* of *half* of an e-transaction belonging to *social* reality (the creation of a debt in compensation of some commodity) with *half* of one belonging to *physical* reality (our firm receiving finished goods from its supplier) – this half-and-half combination illustrates why an a-transaction is an *abstraction* and lacks the reality status of an e-transaction. For further details see Mattessich (1989a,b), pp. 11–31.

[9] I.e. $<$ is a binary relation on T which is transitive, anti-reflexive and connected.
[10] The restriction of ET to t, $ET \cap (\{t\} \times S \times S)$, is obtained from ET by omitting all 'parts' of ET referring to instants different from t.
[11] For further details see Mattessich (1964/77), pp. 452–454.
[12] For the notion of set-theoretic structure see, for example, Balzer, Moulines, and Sneed (1987), Ch. 1.
[13] But the *aggregation problem* must not be confused with the *valuation problem*. The *linear* aggregation in D4–6 is determined by the fact that monetary amounts, by their very nature, possess linear aggregation. This is independent from the valuation problem, because even in accounting non-linear valuation is possible (see Mattessich 1964/77, pp. 224–231) and, in an indirect way, does occur routinely (e.g. through non-linear depreciation). The present notation in D4 deliberately neglects but implies references to entity e for which the account is kept – yet we could, of course, write $a = \langle T, n, R, c, \delta, B_a, e \rangle$ and $B_a(t, e)$.
[14] Cf. Mattessich (1989a), pp. 217–219 for a fuller discussion of the principle of accountability.
[15] Mattessich (1989a), pp. 217–219.
[16] Cf. Balzer, Moulines, and Sneed (1987) for reference to technical terms used in the following.
[17] Ijiri's (1982, 1989) triple-entry bookkeeping is a typical case of a multi-entry system, and justifies the need for a more general formalization of accounting as presented in this section.
[18] Cf. Balzer, Moulines, and Sneed (1987), Chap. IV.
[19] Precisely speaking, D6 cannot be applied here because the d function used in D6 is of a different type than that of g. However, since neither d nor g are here involved explicitly, D6 may be trivially adjusted to the present case.

REFERENCES

American Accounting Association: 1966, *A Statement of Basic Accounting Theory* (Sarasota, FL).

Avila, H.E., Bravo, G. and Scarano E.R.: 1988, 'An Axiomatic Foundation of Accounting', Working Paper, Institute of Accounting Research, University of Buenos Aires.

Balzer, W., Moulines, C.U. and Sneed, J.D.: 1987, *An Architectonic for Science* (Dordrecht: D. Reidel Publ. Co.).

Carlson, M.L. and Lamb J.W.: 1981, 'Constructing a Theory of Accounting: An Axiomatic Approach', *The Accounting Review* (July), 557–573.

Chambers, R.J.: 1966, *Accounting, Evaluation and Economic Behavior* (Englewood Cliffs, N.J.: Prentice-Hall, Inc.)

Deguchi, H. and Nakano B.: 1986, 'Axiomatic Foundations of Vector Accounting', *Systems Research* 3(1), 31–39.

De Pree, C.M.: 1989, 'Testing and Evaluating a Conceptual Framework of Accounting', *ABACUS* (September) 61–84.

Financial Accounting Standards Board: 1976, *Scope and Implications of The Conceptual Framework Project* (Stamford, CT: FASB).

Financial Accounting Standards Board: 1978–80, *Statements of Financial Accounting Concepts* (Stamford, CT: FASB).

Galassi, Giuseppe: 1978, *Sistemi contabili assiomatici e sistemi teorici deduttivi* (Bologna: Patron Editore).

Ijiri, Yuji: 1965, 'Axioms and Structures of Conventional Accounting Measurement', *The Accounting Review* (January), 36–53.

Ijiri, Yuji: 1967, *The Foundations of Accounting Measurement* (Englewood Cliffs, NJ: Prentice-Hall). Reprinted, Houston, Scholars Book Co.

Ijiri, Yuji: 1975, *Theory of Accounting Measurement* (American Accounting Association).

Ijiri, Yuji: 1979, 'A Structure of Multisectional Accounting and Its Applications to National Accounting', *Eric Louis Kohler*, ed. by W. Cooper and Y. Ijiri (Reston, Virg.: Reston Publishing Co.), pp. 208–223.

Ijiri, Yuji: 1982, *Triple-Entry Bookkeeping and Income Momentum* (Sarasota, Fa.: American Accounting Association).

Ijiri, Yuji: 1989, *Momentum Accounting and Triple-Entry Bookkeeping: Exploring the Dynamic Structure of Accounting Measurements* (Sarasota, Fa.: American Accounting Association).

Kosiol, Erich: 1970, 'Zur Axiomatik der Theorie der pagatorischen Erfolgsrechnung', *Zeitschrift Für Betriebswirtschaft* **22**, 135–162.

Krantz, D. H., Luce, R. D., Suppes, P. and Tversky A.: 1971, *Foundations of Measurement* (New York: Academic Press).

Mattessich, Richard: 1957/82, 'Towards a General and Axiomatic Foundation of Accounting – With an Introduction to the Matrix Formulation of Accounting Systems', *Accounting Research* (October), 328–355. Reprinted in *The Accounting Postulates and Principles Controversy of the* 1960*s* ed. by S.A. Zeff (1982), New York: Garland Publ., Inc.

Mattessich, Richard: 1961, 'Budgeting Models and System Simulation', *The Accounting Review* (July), 384–397.

Mattessich, Richard: 1964/77, *Accounting and Analytical Methods – Measurement and Projection of Income and Wealth in the Micro- and Macro-Economy*, Homewood, Ill. Reprinted in facsimile, Ann. Arbor, Mich.: University Microfilms International.

Mattessich, Richard: 1964/79, *Simulation of the Firm through a Budget Computer Program*, Homewood, Ill. Reprinted in facsimile, Ann Arbor, Michigan: University Microfilms International.

Mattessich, Richard: 1970, *Die wissenschaftlichen Grundlagen des Rechnungswesens* (Gütersloh: Bertelsmann Universitätsverlag).

Mattessich, Richard: 1972, 'Methodological Preconditions and Problems of a General Theory of Accounting', *The Accounting Review* (July), 469–487.

Mattessich, Richard: 1973, 'On the Axiomatic Formulation of Accounting: Comment of Professor S. Saito's Considerations'. Originally in Japanese in *Sangyo Keiri*, Tokyo, **33**(3), 70–77, and (4), 71–75. English translation (1973) in *Musashi University Journal*, **21**(1–2), 77–94.

Mattessich, Richard (ed.): 1984/89, *Modern Accounting Research, History, Survey, and Guide*, Vancouver, B.C.: Canadian Certified General Accountants Research Foundation (first, and reprint edition).

Mattessich, Richard: 1987a, 'An Applied Scientist's Search for a Methodological Framework', *Logic, Philosophy of Science and Epistemology* ed. by Paul Weingartner and Gerhard Schurz, Vienna: Verlag Hölder-Pichler-Tempsky.

Mattessich, Richard: 1987b, 'Prehistoric Accounting and the Problem of Representation: On Recent Archaeological Evidence of the Middle-East from 8000 B.C. to 3000 B.C.', *The Accounting Historians Journal*, **14**, 71–91.

Mattessich, Richard: 1989a, 'Counting, Accounting, and the Input-Output Principle' (Award for Best Paper 1988). *Proceedings of 1988 Annual Conference of the Canadian Academic Accounting Association* ed. by Jeffrey Kantor (Windsor, Ont.: CAAA), pp. 199–230.

Mattessich, Richard: 1989b, 'Accounting and the Input–Output Principle in the Prehistoric and Ancient World', *ABACUS* (September), 74–84.

Moonitz, Maurice: 1961/82, *The Basic Postulates of Accounting*, New York: American Institute of Certified Public Accountants. Reprinted in Zeff (1982).

Morgenstern, Oskar: 1963, 'Limits to the Uses of Mathematics in Economics', *Mathematics and the Social Sciences* ed. by J. C. Charlesworth, Philadelphia: American Academy of Political and Social Sciences, pp. 12–29.

Nehmer, Robert A.: 1988, *Accounting Information System as Algebras and First Order Axiomatic Models*, Doctoral Dissertation, Urbana-Champaign: Dept. of Accounting of the University of Illinois.

Orbach, K. N.: 1978, *Accounting as a Mathematical Measurement Theoretic Discipline*, Ph.D. Dissertation. Texas A&M University.

Paton, William W.: 1922/62, *Accounting Theory – With Special Reference to the Corporate Enterprise* (New York: Ronald Press). Reprint edition: Lawrence, Kaus.: Scholars Book Co.

Saito, Shizuko: 1972, 'Some Considerations on the Axiomatic Formulation of Accounting'. Originally in Japanese in *Kaikei*, **101**, 45–65, English translation in *The Mushashi University Journal*, **20**, 81–99.

Saito, Shizuko: 1973, 'Further Considerations on the Axiomatic Formulation of Accounting: A Reply to Professor Mattessich' in the *Mushashi University Journal*, **21**(1–2) 95–107.

Schmandt-Besserat, Denise: 1982, 'The Emergence of Recordings', *American Anthropologist* **84**, 871–878.

Schmandt-Besserat, Denise: 1988, 'Accountancy in Prehistory', *Proceedings of the 5th World Congress of Accounting Historians*, Sydney, Australia: University of Sydney.

Schweitzer, Marcell: 1970, 'Axiomatik des Rechnungswesens', *Handwörterbuch des Rechnungswesens* ed. by E. Kosiol (Stuttgart: Poeschel Verlag) pp. 83–90.

Sprouse, R. and Moonitz, M.: 1962/82, *A Tentative Set of Broad Accounting Principles for Business Enterprise*, (New York: AICPA). Reprinted in Zeff (1982).

Tanaka, Shigetsugu: 1982, *The Structure of Accounting Language* (Tokyo: Chuo University Press).

Tippett, M.: 1978, 'The Axioms of Accounting Measurement', *Accounting and Business Research* (Autumn), 266–278.

Willett, R. J.: 1987, 'An Axiomatic Theory of Accounting Measurement', *Accounting and Business Research* (Spring), 155–171.

Willett, R. J.: 1988, 'An Axiomatic Theory of Accounting Measurement / Part II', *Accounting and Business Research* (Winter), 79–91.

Willett, R. J.: 1989, 'The Measurement of Theoretic and Statistical Foundations of the Transactions Theory of Accounting Numbers', (Aberystwyth: University College of Wales, Working Paper).

Winborn, M.G.: 1962, *Application of Sets and Symbolic Logic to Selected Accounting Principles and Practices*, Ph.D. Dissertation, University of Texas.

Zeff, Stephen A. (ed.): 1982, *The Accounting Postulates and Principles Controversy of the 1960s* (New York, Garland Publ., Inc.).