

Correlation Smile Matching for CDO Tranches with α Stable Distributions and Fitted Archimedean Copulas

Scherer, Wolfgang; Prange, Dirk

Postprint / Postprint

Zeitschriftenartikel / journal article

Zur Verfügung gestellt in Kooperation mit / provided in cooperation with:

www.peerproject.eu

Empfohlene Zitierung / Suggested Citation:

Scherer, W., & Prange, D. (2009). Correlation Smile Matching for CDO Tranches with α Stable Distributions and Fitted Archimedean Copulas. *Quantitative Finance*, 9(4), 439-449. <https://doi.org/10.1080/14697680802464428>

Nutzungsbedingungen:

Dieser Text wird unter dem "PEER Licence Agreement zur Verfügung" gestellt. Nähere Auskünfte zum PEER-Projekt finden Sie hier: <http://www.peerproject.eu>. Gewährt wird ein nicht exklusives, nicht übertragbares, persönliches und beschränktes Recht auf Nutzung dieses Dokuments. Dieses Dokument ist ausschließlich für den persönlichen, nicht-kommerziellen Gebrauch bestimmt. Auf sämtlichen Kopien dieses Dokuments müssen alle Urheberrechtshinweise und sonstigen Hinweise auf gesetzlichen Schutz beibehalten werden. Sie dürfen dieses Dokument nicht in irgendeiner Weise abändern, noch dürfen Sie dieses Dokument für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, aufführen, vertreiben oder anderweitig nutzen.

Mit der Verwendung dieses Dokuments erkennen Sie die Nutzungsbedingungen an.

gesis
Leibniz-Institut
für Sozialwissenschaften

Terms of use:

This document is made available under the "PEER Licence Agreement". For more Information regarding the PEER-project see: <http://www.peerproject.eu>. This document is solely intended for your personal, non-commercial use. All of the copies of this documents must retain all copyright information and other information regarding legal protection. You are not allowed to alter this document in any way, to copy it for public or commercial purposes, to exhibit the document in public, to perform, distribute or otherwise use the document in public.

By using this particular document, you accept the above-stated conditions of use.

Mitglied der

Leibniz-Gemeinschaft



Correlation Smile Matching for CDO Tranches with α Stable Distributions and Fitted Archimedean Copulas

Journal:	<i>Quantitative Finance</i>
Manuscript ID:	RQUF-2006-0086.R3
Manuscript Category:	Research Paper
Date Submitted by the Author:	08-May-2008
Complete List of Authors:	Scherer, Wolfgang; Dresdner Kleinwort, RMT Model Validation Credit Derivatives Prange, Dirk; Citigroup, FICC Quantitative Credit Analysis
Keywords:	Credit Derivatives, Copulas, Correlation, Credit Models
JEL Code:	C00 - General < C0 - General < C - Mathematical and Quantitative Methods, C16 - Specific Distributions < C1 - Econometric and Statistical Methods: General < C - Mathematical and Quantitative Methods
Note: The following files were submitted by the author for peer review, but cannot be converted to PDF. You must view these files (e.g. movies) online.	
Prange_Scherer_AlphaStableCopula_3rdRevision_2008-05-08.tex RQUF-2006-0086.R3.ZIP	

SCHOLARONE™
Manuscripts

Correlation Smile Matching for CDO Tranches with α -Stable Distributions and Fitted Archimedean Copula Models

Dirk Prange *
Citigroup
FICC
Credit Quantitative Analysis

Wolfgang Scherer †
Dresdner Kleinwort
Risk Methodology Trading
Model Validation Credit Derivatives

21 November 2006, 1st revision 28 August 2007, 2nd revision 08 May 2008

Abstract

As an extension of the standard Gaussian copula model to price CDO tranche swaps we present a generalization of a one-factor copula model based on stable distributions. For special parameter values these distributions coincide with Gaussian or Cauchy distributions, but changing the parameters allows a continuous deformation away from the Gaussian copula. All these factor copulas are embedded into a framework of stochastic correlations.

We furthermore generalize the linear dependence in the usual factor approach to a more general Archimedean copula dependence between the individual trigger variable and the common latent factor.

Our analysis is carried out on a non-homogeneous correlation structure of the underlying portfolio. CDO tranche market premia, even throughout the correlation crisis in May 2005, can be reproduced by certain models. From a numerical perspective all these models are simple since calculations can be reduced to one-dimensional numerical integrals.

1 Introduction

Over the recent years, dependence modeling for basket type credit derivatives has evolved along with the increase in trading volume of such instruments. Among these, Collateralized Debt Obligations (CDOs) and their tranches still pose a modeling challenge when trying to reproduce the market observed prices.

After a first approach via binomial expansion techniques and other modeling attempts, copula based models have become more widely discussed and used for CDOs. Perhaps the historical starting point was the by now standard Gaussian copula of Li [1]. Since then, various modifications and extensions thereof have been proposed. In order to replicate the market observed correlation smile, Andersen and Sidenius [2] investigated extending the Gaussian one-factor copula model to include random recovery and/or making the factor loading state dependent as well, thus introducing what has become known as *local correlation*, while still keeping the Gaussian distribution of the latent and idiosyncratic random factors. While

*Email: Dirk.Prange@citi.com

†Email: Wolfgang.Scherer@dkib.com

both random recovery and local correlation do induce some correlation smile, this does not appear to be sufficiently pronounced to always replicate the full market. In a similar spirit Burtschell et al. [3] have extended the Gaussian copula with state dependent correlation to produce a market skew.

Instead of generalizing the factor loadings, Kalemanova et al. [4] replace the Gaussian copula by the copula generated from normal inverse Gaussian distributed factors. Their analysis, however, was done using the large homogenous portfolio approximation, rather than being based on the true individual characteristics of the underlying portfolio. As such their approach is not suitable to provide a basis for consistently pricing sub-portfolios or delivering sensitivities w.r.t. individual names. A recent comparison of different copula models such as Student, Double Student, Clayton, and Marshall-Olkin copulas is given in [5].

An approach not based on factor copulas, but on generalized Archimedean copulas, has been presented by Rogge and Schönbucher [6], but while this is an interesting idea, which can be generalized even further, it is not clear at present which, if any, of the generalizations actually reproduces the observed market tranche prices. Recent work by Hull and White [7] also takes the direction of generalizing the underlying copula by implying out the distribution of the hazard rate path from observed market prices.

In this paper we present several approaches based on a combination and extension of various of the ideas mentioned above and investigate their suitability to reproduce the tranche market for several dates in 2005, with particular emphasis on the period of the correlation crisis in May.

We first consider a class of one-factor models based on α -stable distributions for the factors. This gives rise to α -stable copulas, which include the Gaussian copula as a special case, but allow for continuous deformation away from the Gaussian case to ones where the factor distributions have increasingly fatter tails. The parameters of the α -stable distribution can be determined by calibration to the tranche market and provide a good fit across the full capital structure. This is along the lines of investigations making use of non-Gaussian factor distributions which have also been followed by Albrecher et. al. [8], Moosbrucker [9], Baxter [10], and Guegan and Houdain [?].

Calibration to market can be further enhanced, if these copulas are combined with the above copula mixing via state dependent correlations as outlined in [3]. This results in a near perfect fit under normal market conditions and a very good fit for markets during the correlation crisis. As shown in this paper, these α -stable one-factor models with stochastic correlations calibrate well to the market observed tranche quotes.

Apart from the α -stable copula, we also present two other generalizations of the one-factor modeling approach. These are based on a generalization of the dependence structure of the usual one-factor approach. Their calibration performance will be compared to that of the α -stable copula for the same markets. To our knowledge, they have not been discussed in the literature so far and can be summarized as follows.

In what can be viewed as a further generalization of local correlations, we generalize the factor dependence from merely linear coupling to a two-dimensional coupling between latent variable and common driver, based on a copula mixture, including a number of Archimedean copulas.

Lastly, we introduce a piecewise linear generator function for an Archimedean copula such that the interpolation points for that generator are determined from observed tranche market data. This is somewhat akin to Hull and White's search for the "Perfect Copula".

The paper is organized as follows: In Section 2 we summarize the basic assumptions un-

derlying one-factor models based on conditional independence. We introduce the extension to stochastic correlation in Section 2.2. α -stable distributions, their properties and the resulting α -stable copulas are discussed in Section 2.3. In Section 2.4 we generalize the linear dependence of the latent factor to a two dimensional copula which can be either a mixture of copulas or an Archimedean copula with a generalized piecewise linear generator. In Section 3 we present the calibration procedure and the results for the iTraXX tranche market from 26 April 2005 until 07 June 2005 which arguably comprises a more difficult period for the tranche market. The paper then ends with a concluding section.

2 Default Dependencies

2.1 Basic Assumptions and Resulting Dependence Structures

We consider a portfolio $U := \{1, \dots, u\}$ of u underlyings, where $i \in U$ refers to the i th underlying name. The default time of underlying i is denoted by τ_i . The following assumptions form the basis of all conditional independence loss models:

1. A default of the i th underlying taking place on or before t is equivalent to a random variable X_i falling below a barrier $b_i(t)$, i.e.

$$\{\tau_i \leq t\} = \{X_i \leq b_i(t)\} \quad (1)$$

which has as a consequence

$$P_i(t) := \mathbb{P}\{\tau_i \leq t\} = \mathbb{P}\{X_i \leq b_i(t)\} = F_{X_i}(b_i(t)) \quad (2)$$

i.e.

$$b_i(t) = F_{X_i}^{-1}(P_i(t)) \quad (3)$$

where $F_A(z) := \mathbb{P}\{A \leq z\}$ denotes the the *cumulative distribution function* of a random variable A .

An alternative way of stating this, which perhaps makes the probabilistic construction of default times more explicit, is

$$\tau_i = P_i^{-1}(F_{X_i}(X_i)). \quad (4)$$

Eqn. (4) also shows that each default time τ_i is a non-decreasing function of X_i .

2. The variables X_i are given by a common risk driver (the *latent* variable) X and idiosyncratic risks \bar{X}_i such that

$$X_i = c_i X + \bar{c}_i \bar{X}_i \quad i \in U \quad (5)$$

where $X, \bar{X}_1, \dots, \bar{X}_u$ are independent random variables and all \bar{X}_i have the same distribution.

The common risk driver X can be one- or d -dimensional (with $d > 1$), resulting in a so-called one- or d -factor Model. Here we concentrate on the one-factor case, but this can easily be generalized¹.

¹Although it is far from clear, whether more factors will always improve the model.

Note that it is the joint distribution of the X_i which gives rise to the joint distribution of defaults via Eqn. (1) (see also Eqn. (12) below). The introduction of X and the \bar{X}_i and the linear relation Eqn. (5) is only one (albeit comfortable) way to induce a dependence structure. In the case where the second moments of X_i exist one finds from Eqn. (5) that

$$\text{Cov}(X_i, X_j) = c_i c_j \text{Var}(X) + \delta_{ij} \bar{c}_i^2 \text{Var}(\bar{X}_i) \quad (6)$$

where δ_{ij} denotes the Kronecker delta, which is $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. As a consequence of Eqn. (6) the values of the c_i are often interpreted as determining “a correlation” in the portfolio. For the α -stable distributions considered below, these second (or higher) moments (i.e. the right hand side of Eqn. (6)) no longer exist. Nevertheless these models give rise to strong dependence features, which now, however, are not determined by the c_i alone.

Moreover, we can replace the linear relation in Eqn. (5) by something more general. Hence, in Section 2.4 below we will generalize the *linear* dependence structure encoded in Eqn. (5) to a dependence structure between X and X_i originating from a two-dimensional copula.

It follows from Eqn. (5) that

$$P_i(t, x) := \mathbb{P}\{\tau_i \leq t | X = x\} = F_{\bar{X}_i} \left(\frac{F_{X_i}^{-1}(P_i(t)) - c_i x}{\bar{c}_i} \right) \quad (7)$$

and that, conditional on X , the defaults of the underlyings are independent. Eqn. (7) also allows to calculate the default density

$$p_i(t, x) := \mathbb{P}\{\tau_i \in [t, t + dt] | X = x\} = \frac{dP_i(t, x)}{dt} = f_{\bar{X}_i} \left(\frac{F_{X_i}^{-1}(P_i(t)) - c_i x}{\bar{c}_i} \right) \frac{p_i(t)}{\bar{c}_i f_{\bar{X}_i}(F_{X_i}^{-1}(P_i(t)))} \quad (8)$$

where $f_A(z) := \frac{d}{dz} F_A(z)$ denotes the *probability density function* of a random variable A .

The conditional independence can then be used to calculate the full default (and thus loss) distribution as

$$\begin{aligned} \mathbb{P}\{\tau_1 \leq t_1, \dots, \tau_u \leq t_u\} &= \mathbb{E}[\mathbb{P}\{X_1 \leq b_1(t_1), \dots, X_u \leq b_u(t_u) | X = x\}] = \mathbb{E} \left[\prod_{i \in U} P_i(t_i, X) \right] \\ &= \int \prod_{i \in U} P_i(t_i, x) dF_X(x) \end{aligned} \quad (9)$$

and similarly for the default density. This distribution is required to calculate payoffs which depend on the identity of the defaulted obligor, like Nth-To-Default Swaps (NTD) with different recoveries.

To obtain the portfolio loss up to time t , which we denote by $L(t)$, it is, however, simpler to use by now well-known recursive methods as e.g. outlined in [2]. Suppose then, that $\mathbb{P}\{L(t) = Z | X = x\}$ has been obtained by such a recursive procedure. This yields the full (unconditional) loss distribution at time t as

$$\mathbb{P}\{L(t) = Z\} = \int \mathbb{P}\{L(t) = Z | X = x\} dF_X(x) \quad (10)$$

and this distribution can be used to calculate expected payoffs for derivatives depending on the losses in a portfolio such as e.g. NTDs with homogeneous recoveries or CDO portfolio tranche swaps.

Let

$$F_{X_U}(a_1, \dots, a_u) := \mathbb{P}\{X_1 \leq a_1, \dots, X_u \leq a_u\}$$

denote the *joint distribution* of the $X_U := (X_1, \dots, X_u)$ and

$$C_{X_U}(a_1, \dots, a_u) := F_{X_U}(F_{X_1}^{-1}(a_1), \dots, F_{X_u}^{-1}(a_u)) \quad (11)$$

their *copula*. It follows from their respective definitions and from Eqn. (1) and Eqn. (3) that this is *equal to the copula of default times*

$$\begin{aligned} C_{\tau_U}(a_1, \dots, a_u) &:= F_{\tau_U}(F_{\tau_1}^{-1}(a_1), \dots, F_{\tau_u}^{-1}(a_u)) \\ &= \mathbb{P}\{\tau_1 \leq P_1^{-1}(a_1), \dots, \tau_u \leq P_u^{-1}(a_u)\} \\ &= \mathbb{P}\{X_1 \leq b_1[P_1^{-1}(a_1)], \dots, X_u \leq b_u[P_u^{-1}(a_u)]\} \\ &= \mathbb{P}\{X_1 \leq F_{X_1}^{-1}(a_1), \dots, X_u \leq F_{X_u}^{-1}(a_u)\} \\ &= C_{X_U}(a_1, \dots, a_u), \end{aligned} \quad (12)$$

which is a consequence of the fact that the default times τ_i are monotonic functions of the X_i , as seen from Eqn. (4).

In the first generation copula model [1] the joint distribution F_{X_U} on the right hand side of Eqn. (11) was chosen to be a multidimensional Gaussian distribution, i.e. $F_{X_U} = \Phi_{\rho}^{Gauss}$ with a given correlation matrix ρ , thus leading in Eqn. (12) to the Gaussian copula $C_{X_U} = C_{Gauss}$. Since then other choices for F_{X_U} (see e.g. [5]) have been investigated as well.

2.2 Incorporating stochastic correlation

Since the conditionally independent Gaussian model does not provide any further parameters, its ability to fit to observed market tranche premia for all tranches is insufficient, such that the observed premia can only be replicated with different correlation inputs for each tranche, giving rise to what is now referred to as the correlation smile. Burtschell et al. [3] therefore considered to take stochastic correlations into account. We adopt their approach here, allowing the dependence to be a mixture of co-monotonicity, independence and the original conditionally independent model. Therefore, we introduce two Bernoulli variables B_a, B_b , which are independent and independent of the factors \bar{X}_i, X . Their probabilities are denoted by $q_l = \mathbb{P}\{B_l = 1\}$, $l = a, b$, and we set

$$\begin{aligned} \tilde{X}_i &= (1 - B_a)(1 - B_b)X_i + B_b X + (1 - B_b)B_a \bar{X}_i \\ &= [c_i(1 - B_a)(1 - B_b) + B_b]X + [\bar{c}_i(1 - B_a) + B_a](1 - B_b)\bar{X}_i \end{aligned} \quad (13)$$

which implies

$$\tilde{X}_i = \begin{cases} X, & \text{if } B_b = 1 \\ \bar{X}_i, & \text{if } B_b = 0 \text{ and } B_a = 1 \\ c_i X + \bar{c}_i \bar{X}_i, & \text{if } B_b = 0 \text{ and } B_a = 0. \end{cases} \quad (14)$$

The case $B_b = 1$ thus corresponds to complete dependence for the \tilde{X}_i , whereas the case $B_b = 0$ and $B_a = 1$ corresponds to full independence and lastly the case $B_b = 0$ and $B_a = 0$

corresponds to the standard copula situation. A somewhat related setup has also been used by Tavares et al. in [11].

The default event for name i is now triggered by \tilde{X}_i falling below a barrier, i.e.

$$\{\tau_i \leq t\} = \{\tilde{X}_i \leq b_i(t)\}. \quad (15)$$

The conditional probabilities are given by

$$\begin{aligned} \mathbb{P}\left\{\tilde{X}_i < z \mid X = x\right\} &= \sum_{k=0}^1 \sum_{l=0}^1 \mathbb{P}\left\{\tilde{X}_i < z \mid X = x \wedge B_a = k \wedge B_b = l\right\} \mathbb{P}\{B_a = k\} \mathbb{P}\{B_b = l\} \\ &= \mathbf{I}_{\{x \leq z\}} q_b + F_{\bar{X}_i}(z) q_a (1 - q_b) + (1 - q_a)(1 - q_b) F_{\bar{X}_i}\left(\frac{z - c_i x}{\bar{c}_i}\right) \end{aligned} \quad (16)$$

where

$$\mathbf{I}_{\{A\}} := \begin{cases} 1, & \text{if event } A \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

denotes the indicator function. Integrating out the latent variable X in Eqn. (16) leaves us with

$$\mathbb{P}\left\{\tilde{X}_i \leq z\right\} = q_b F_X(z) + q_a (1 - q_b) F_{\bar{X}}(z) + (1 - q_a)(1 - q_b) F_{X_i}(z) \quad (17)$$

In what follows, we choose the distributions for X , \bar{X}_i and thus X_i such that $F_X = F_{\bar{X}_i} = F_{X_i}$. Note that then Eqn. (17) implies $F_{\tilde{X}_i} = F_{X_i} = F_X$.

2.3 α -Stable Copulas

To obtain a versatile copula structure, we choose the latent and idiosyncratic factors from the following α -stable distributions (using the notation of NOLAN in [12]) $X, \bar{X}_i \sim S(\alpha, \beta, \gamma, \delta; 1)$. Except for certain α values (see below) these distributions cannot be expressed with known functions, but it is only their characteristic function which can be given explicitly as follows

$$\chi_{S(\alpha, \beta, \gamma, \delta; 1)}(x) = \begin{cases} \exp\left(-\gamma^\alpha |x|^\alpha [1 - i\beta(\tan \frac{\pi\alpha}{2}) \operatorname{sgn}(x)] + i\delta x\right), & \text{if } \alpha \neq 1 \\ \exp\left(-\gamma |x| [1 + i\beta \frac{2}{\pi} \operatorname{sgn}(x) \log |x|] + i\delta x\right), & \text{if } \alpha = 1. \end{cases} \quad (18)$$

The parameter α takes values in $]0, 2]$ and its decrease leads to *fatter tails*. The parameter $\beta \in [-1, 1]$ defines the *skewness* of the distribution, while $\gamma \in \mathbb{R}_+$ is a scale and $\delta \in \mathbb{R}$ is a location parameter. For our purposes the scale and location parameters γ and δ can be set to some convenient values, since other scale and location only change the intermediate b_i functions. The term α -stable refers to the fact that the sum of two α -stable random variables is again an α -stable random variable, albeit with possibly different (skewness, scale, or location) parameters. This property which sometimes is also referred to as “summation stability”, can be stated as follows. If for $a = 1, 2$

$$X_a \sim S(\alpha, \beta_a, \gamma_a, \delta_a; 1) \quad (19)$$

and² $c_a \geq 0$ then it follows that

$$c_1 X_1 + c_2 X_2 \sim S(\alpha, \beta, \gamma, \delta; 1) \quad (20)$$

²These distributions are also well defined for $c_a \in \mathbb{R}$ but this only complicates the formulas and is not necessary for our purposes.

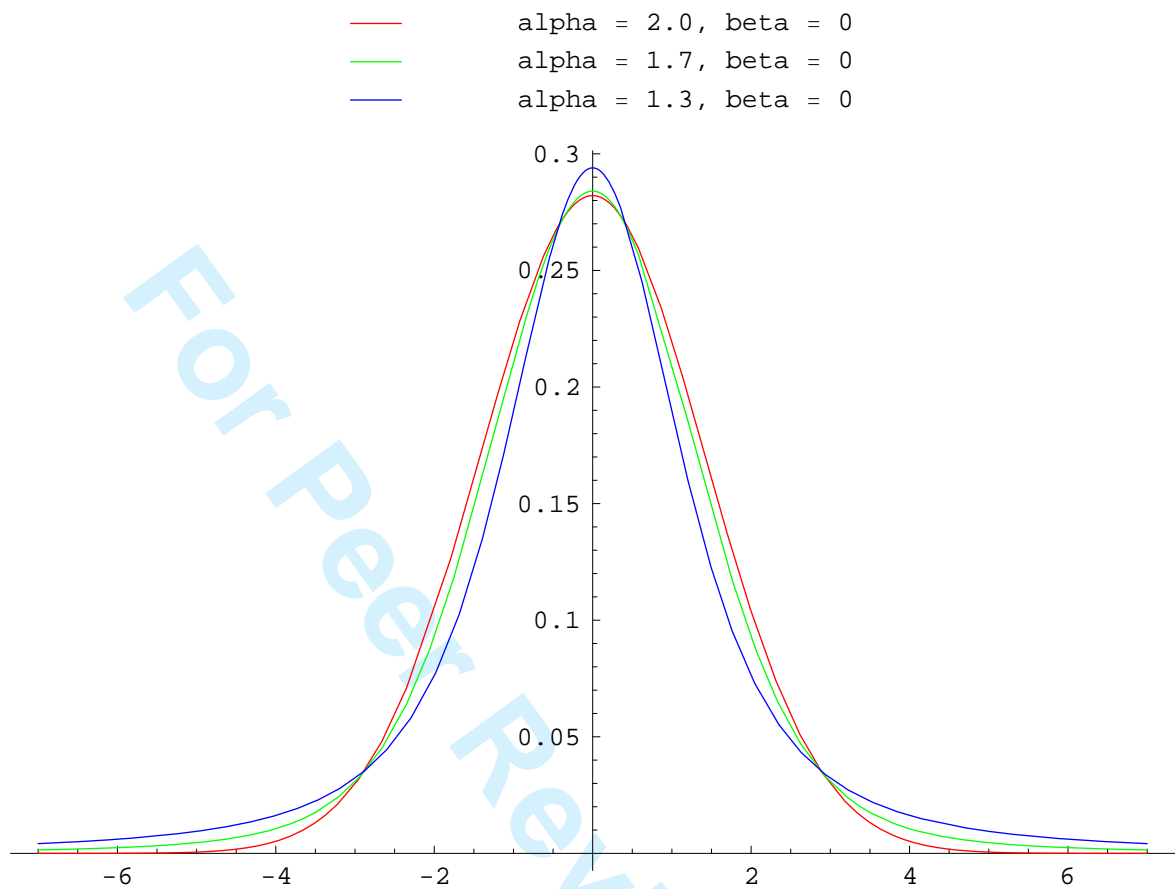


Figure 1: α -stable distributions with different values for α . The value $\alpha = 2$ is the Gaussian distribution. Fatter tails for $\alpha < 2$ are clearly visible.

where

$$\begin{aligned}\beta &:= \frac{\beta_1(c_1\gamma_1)^\alpha + \beta_2(c_2\gamma_2)^\alpha}{(c_1\gamma_1)^\alpha + (c_2\gamma_2)^\alpha} \\ \gamma &:= [(c_1\gamma_1)^\alpha + (c_2\gamma_2)^\alpha]^{\frac{1}{\alpha}} \\ \delta &:= c_1\delta_1 + c_2\delta_2.\end{aligned}$$

Note that second (and higher) moments of such distributions exist only for $\alpha = 2$, which corresponds to a Gaussian distribution. Changing the α parameter, allows us to change the distribution away from the Gaussian case $\alpha_{GAUSS} = 2$ in a smooth fashion to other distributions which have ever increasing fatter tails as α decreases away from $\alpha_{GAUSS} = 2$. A value for $\alpha \in]0, 2]$ may thus be obtained by *calibration*. Figures 1 and 2 show the distribution functions for different values of α and β .

Choosing the scale and location parameters as $\gamma = 1, \delta = 0$, the common risk driver X

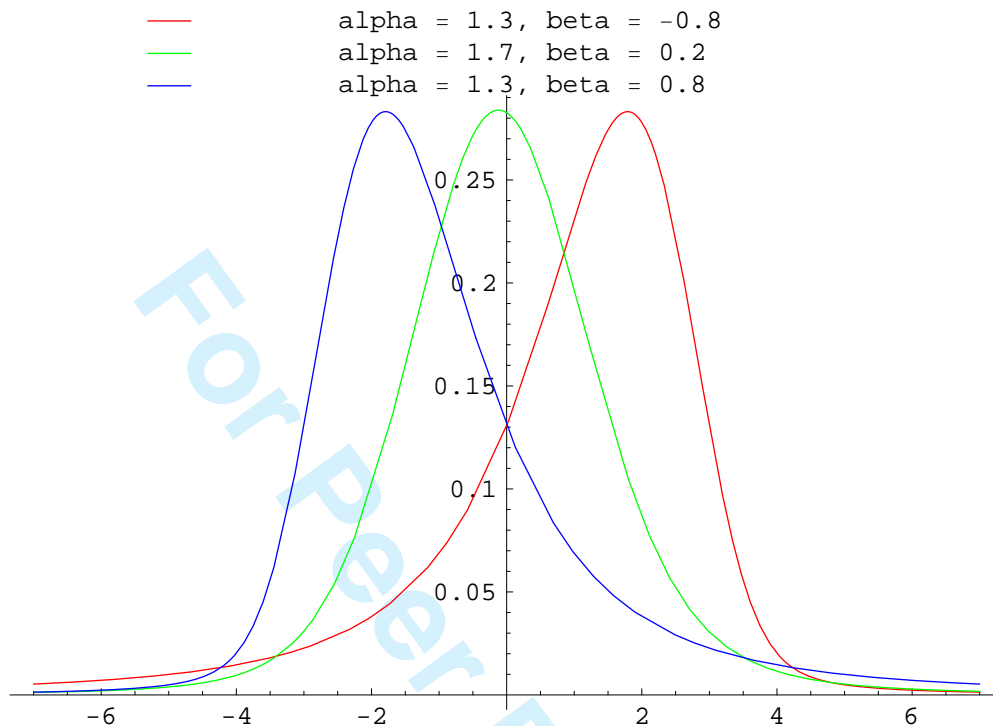


Figure 2: α -stable distributions with different values for α and β . Changing β away from $\beta = 0$ introduces skewness in the distribution.

and the idiosyncratic risks \bar{X}_i are then taken from the α -stable distributions

$$X \sim S(\alpha, \beta, 1, 0; 1) \quad (21)$$

$$\bar{X}_i \sim S(\alpha, \beta, 1, 0; 1) \quad (22)$$

and used as in Eqn. (5) together with the following choice of the coefficients $c_i \in [0, 1]$ and $\bar{c}_i = (1 - c_i^\alpha)^{\frac{1}{\alpha}}$, such that the random variable

$$X_i = c_i X + (1 - c_i^\alpha)^{\frac{1}{\alpha}} \bar{X}_i \sim S(\alpha, \beta, 1, 0; 1) \quad (23)$$

is again an α -stable distributed random variable with the same α, β and $\gamma = 1, \delta = 0$. Denoting the α -stable cumulative probability distribution function by F_α , the conditional default probabilities are then given by

$$P_i(t, x) = F_\alpha \left(\frac{F_\alpha^{-1}(P_i(t)) - c_i x}{(1 - c_i^\alpha)^{\frac{1}{\alpha}}} \right). \quad (24)$$

The stable distribution functions are not explicitly given through combinations of known functions. Therefore numerical methods have to be used in an implementation. One possible solution would be to apply Fast-Fourier Transformations (FFT) on (18). Alternatively, there are numerical libraries available which provide for an implementation of the functions $F_\alpha, F_\alpha^{-1}, f_\alpha$.³

³We have chosen the STABLE library by RobustAnalysis for this.

In the following we will denote the models tested from this group as follows:

STABLE A stable model without stochastic correlation. The free model parameters are (c_i, α, β) .

STABLEmix A stable model with stochastic correlation. The free model parameters are $(q_a, q_b, c_i, \alpha, \beta)$.

The Gaussian copula is obtained as a special subcase, if in Eqn. (21) and in Eqn. (22) the values $\alpha = 2$ and $\beta = 0$ are used. The conditional default probabilities are then given by

$$P_i(t, x) = \Phi \left(\frac{\Phi^{-1}(P_i(t)) - c_i x}{(1 - c_i^2)^{\frac{1}{2}}} \right). \quad (25)$$

where Φ denotes the standard cumulative normal distribution function.

For comparison we include the Gaussian case as a separate model denoted as

GAUSSmix A Gaussian model with stochastic correlation. The free model parameters are (q_a, q_b, c_i) .

Choosing $\alpha = 1$ and $\beta = 0$ in Eqn. (21) and in Eqn. (22) gives a special case of the Cauchy distribution,⁴ which we denote by F_C

$$F_C(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}, \quad (26)$$

The inverse of F_C is explicitly given by: $F_C^{-1}(x) = \tan \left(\pi \left[x - \frac{1}{2} \right] \right)$, and the conditional default probability becomes:

$$P_i(t, x) = \frac{1}{\pi} \arctan \left(\frac{\tan \left(\pi \left[P_i(t) - \frac{1}{2} \right] \right) - c_i x}{1 - c_i} \right) + \frac{1}{2}. \quad (27)$$

In order to compare, we also include the Cauchy case as a separate model denoted as

CAUCHYmix A Cauchy model with stochastic correlation. The free model parameters are (q_a, q_b, c_i) .

2.4 Generalizing Latent Variable Dependence to Copula

In the usual conditional loss setup a linear relation between X_i and the latent variable X (global factor) as given in Eqn. (5) is assumed. This has already been extended in the random factor loading approach [2] and the stochastic correlation approach [3]. Here we generalize this further in that, instead of using a linear coupling in Eqn. (5), a general dependence of X and X_i given through a 2-dimensional copula function $C : [0, 1]^2 \mapsto [0, 1]$ can be assumed. The auxiliary random variables \bar{X}_i are not needed in this case.

Hence, we specify a 2-dimensional copula C , which gives the joint distribution of the variables

$$X, X_i \sim U[0, 1], \quad (28)$$

⁴The full parameterized class of Cauchy distributions can be obtained as special cases of α -stable distributions, but is not needed for our copula modeling here.

such that

$$F_{X_i}(x) = x \quad (29)$$

$$F_X(y) = y \quad (30)$$

$$\mathbb{P}\{X_i \leq x, X \leq y\} = C(x, y) \quad (31)$$

and the corresponding default barriers $b_i(t)$ are given by

$$b_i(t) = \mathbb{P}\{X_i \leq b_i(t)\} = \mathbb{P}\{\tau_i \leq t\} = P_i(t). \quad (32)$$

Like in the previous Section 2.1, X_i and X_j , conditional on X , are independent. Their distribution conditional on X is given by:

$$\mathbb{P}\{X_i \leq z | X = x\} = \frac{\partial}{\partial x} C(z, x) =: \partial_2 C(z, x) \quad (33)$$

such that now

$$P_i(t, x) = \mathbb{P}\{\tau_i \leq t | X = x\} = \mathbb{P}\{X_i < b_i(t) | X = x\} = \partial_2 C(P_i(t), x). \quad (34)$$

The joint default time distribution of the τ_i is still given by (9) with $dF_X(x) = \mathbf{I}_{\{0 < x < 1\}} dx$ being the uniform measure:

$$C(x_1, \dots, x_n) = \int_0^1 dy \prod_{i=1}^n \partial_2 C_i(x_i, y). \quad (35)$$

This can be regarded as a u -dimensional generalization of the copula \star -product

$$C_1 \star \tilde{C}_2(x, y) = \int_0^1 dt \partial_2 C_1(x, t) \partial_1 \tilde{C}_2(t, y)$$

with $\tilde{C}_2(x, y) = C_2(y, x)$ see e.g. [13][chapter 6.3]. Since copulas are continuous their first derivative is non-singular and the right hand side of Eqn. (35) is well defined.

In this copula setup we investigate two approaches:

- **Mixture Copula:** Any convex combination of copula functions will again be a copula. We choose as base functions the Frechet–Hoeffding bounds, the independent copula and an Archimedean copula C_ϕ with lower generator ϕ and form the combination:

$$C(x, y) = w_1 C_\phi(x, y) + w_2 \Pi(x, y) + w_3 M(x, y) + w_4 W(x, y), \text{ with } \sum_{i=1}^4 w_i = 1$$

and where

$C_\phi(x, y) = \phi^{-1}(\phi(x) + \phi(y))$ is an Archimedean copula with generator ϕ ,

$\Pi(x, y) = xy$ the independent copula,

$M(x, y) = \min(x, y)$ the co-monotone copula,

$W(x, y) = \max(x + y - 1, 0)$ the counter-monotone copula.

There is a lot of freedom for the choice of the generator function in this mixture copula. From historical and implied data several authors have found a trend for increasing

correlation in a bad market environment, which in this setup corresponds to a small value of X . We therefore look for an associated copula with positive lower tail dependence. Following the notation of Nelsen [13][table 4.1] we use copulas 1, 16, 19 and 20 of which the last three gave the most convincing results.

Here we report the figures for the Nelsen copula no. 20 which has generator function $\phi_\theta(t) = \exp(t^{-\theta}) - \exp(1)$, $\theta \in (0, \infty)$. For this model we investigate two possibilities:

COPmixHOM A copula mixture model where the Archimedean copula has generator $\phi_\theta(x) = \exp(x^{-\theta}) - \exp(1)$ and flat correlation structure. The free model parameters are (w_1, w_2, w_3, θ) .

COPmixINH An inhomogeneous copula mixture model where the Archimedean copula has the issuer specific generator $\phi_{\theta_i}(x) = \exp(x^{-\theta_i}) - \exp(1)$ leading to a non-flat correlation. The free model parameters are thus $(w_1, w_2, w_3, \theta_i)$.

- **Archimedean copula with non-strict piecewise linear generator:** On a set of nodes $0 = x_0 < x_1 < \dots < x_n = 1$ we specify y values $1 = y_0 > y_1 > \dots > y_n = 0$, and the generator ϕ is defined to be a piecewise linear convex function with $\phi(x_i) = y_i$. The points $y_i, i = 1, \dots, n-1$ must be chosen to satisfy the convexity of ϕ . This model is denoted as

ARCHIMpl A homogenous mixture copula with piecewise linear generator. The free model parameters are the abscissa points x_i and the corresponding function values y_i . Together they are used to construct the linearly interpolated grid: $(x_0, \dots, x_n; y_0, \dots, y_{n-1})$, where $y_n = \phi(x_n) = \phi(1) = 0$ by construction.

3 Calibration

3.1 Reducing the number of parameters

For the models to provide real world prices, a choice of parameters and functions needs to be made. We have chosen to determine the parameters by *multi-dimensional calibration* as follows.

- q_a, q_b in all mixture models GAUSSmix, CAUCHYmix, STABLEmix
- α, β in all α -stable models STABLE, STABLEmix
- θ, w_1, w_2, w_3 in the copula mixture models COPmixHOM, COPmixINH
- y_1, y_2, y_3, y_4 in the generalized Archimedean copula model ARCHIMpl

It remains, to determine the c_i for GAUSSmix, CAUCHYmix, STABLEmix, STABLE, which we consider next and the θ_i for COPmixINH, which we consider thereafter. Both will add one more parameter λ to be calibrated.

In many applications of the conditional default models the c_i from Eqn. (5) are set to the same value, which will be determined by a calibration procedure (e.g. base correlation). We prefer, however, to retain more specific information as to how each issuer is coupled to the economy, hence we do not set all c_i equal. Rather than determining the c_i directly as

input (e.g. from KMV data), we apply a one-parameter approach that allows to retain the relative issuer specific character, but at the same time is amenable to a calibration procedure. Hence, we take couplings $\beta_i \in [0, 1]$ from a database (e.g. KMV) which should already express the dependence of obligor i on a joint global factor. In the spirit of transforming historical market parameters into implicit ones we then apply a global scaling of these by setting $c_i = \beta_i^\lambda, i = 1, \dots, u$ and $\lambda \in [0, \infty)$, such that for $\lambda > 1$ it follows that $c_i < \beta_i$ and $\lambda < 1$ implies $c_i > \beta_i$. Moreover, $\lim_{\lambda \rightarrow \infty} c_i = 0$ and $\lim_{\lambda \rightarrow 0} c_i = 1$. In such an approach we take the variability of the couplings into account while keeping their ordering. The boundaries $\beta_i = 0$ (corresponding to independence) and $\beta_i \rightarrow 1$ (corresponding to co-monotonicity) remain preserved under this scaling. The single parameter λ is obtained within the multi-dimensional calibration.

In order to take the inhomogeneity of the dependence for COPmixINH into account, we chose $\theta_i = -\lambda \log(1 - \beta_i)$ with $\lambda \in [0, \infty)$. Here β_i is again obtained from some database (as e.g. KMV) such that it reflects the dependence of obligor i on a common economic wide factor. As before $\beta_i = 0$ corresponds to independence, $\beta_i \rightarrow 1$ corresponds to co-monotonicity, and the single parameter λ is then obtained within the multi-dimensional calibration.

The inhomogeneity in the couplings is no necessity for the calibration quality of the models. We have tested GAUSSmix, CAUCHYmix and STABLEmix also for a single coupling parameter $c_i = c, \forall i$, producing the same quality of fit like their inhomogeneous counterparts.

3.2 Calibration procedure and results

In applying the model to the pricing of CDO tranche swaps, we focused on the iTraxx series. During May 2005 the so-called correlation crisis led to high equity tranche spreads while the lowest mezzanine kept relatively stable. We investigate the capability to calibrate the models to weekly quotes starting from 26 April until 07 June.

The calibration is performed by first bootstrapping all 125 individual default probability curves and using these as inputs for the models. All swaps (single name and tranche) are quarterly and use Act/360 day-count for the premium leg. Recovery rates are set to 40% for the single names as well as for the iTraxx loss payments. Since the index does not trade on the theoretical level, which is completely determined by the 125 default probability curves, the single name curves are simultaneously adjusted to match the quoted index level.

Parameters for the various models are found by using a multidimensional minimization routine.⁵

Below we will thus consider the following models for their suitability to match a tranche market

GAUSSmix A Gaussian model with stochastic correlation. The model parameters to be calibrated are (q_a, q_b, λ) .

CAUCHYmix A Cauchy model with stochastic correlation. The model parameters to be calibrated are (q_a, q_b, λ) .

STABLEmix A stable model with stochastic correlation. The model parameters to be calibrated are $(q_a, q_b, \lambda, \alpha, \beta)$.

⁵We use a Levenberg Marquardt algorithm on weighted square deviations.

In case of convergence, the solver usually finds a minimum in less than 50 steps, which on an average pc takes between 30s for a Gaussian Model with 2 parameters and 2 min for a stable model with 5 parameters.

STABLE A stable model without stochastic correlation. The model parameters to be calibrated are (λ, α, β) .

COPmixHOM A copula mixture model where the Archimedean copula has generator $\phi_\theta(x) = \exp(x^{-\theta}) - \exp(1)$ and flat correlation. Instead of finding the convex coefficients w_i directly we parameterized them in form of spherical coordinates ψ, θ, ϕ according to

$$\begin{aligned} w_1 &= (\cos \psi_a)^2, \\ w_2 &= (\sin \psi_a \cos \psi_b)^2, \\ w_3 &= (\sin \psi_a \sin \psi_b \cos \psi_c)^2, \\ w_4 &= (\sin \psi_a \sin \psi_b \sin \psi_c)^2 \end{aligned}$$

The model parameters to be calibrated are $(\psi_a, \psi_b, \psi_c, \theta)$. In the table of results given below the parameters are found in the following columns: $\psi_a = q_a, \psi_b = q_b$.

COPmixINH An inhomogeneous copula mixture model where the Archimedean copula has the issuer specific generator $\phi_{\theta_i}(x) = \exp(x^{-\theta_i}) - \exp(1)$ where $\theta_i = \lambda \left(\frac{1}{1-\beta_i} - 1 \right)$. The model parameters to be calibrated are $(\psi_a, \psi_b, \psi_c, \lambda)$ and the w_i are parameterized as for COPmixHOM.

ARCHIMpl A homogenous mixture copula with piecewise linear generator. The abscissa points are chosen to be

$$\begin{aligned} x_0 &= 0.000, & x_1 &= 0.003, & x_2 &= 0.010, & x_3 &= 0.030, \\ x_4 &= 0.050, & x_5 &= 0.0150, & x_6 &= 1.000. \end{aligned}$$

Since an overall scaling of the generator function $\phi \rightarrow \lambda \phi, \lambda \in \mathbb{R}^+$, leads to the same copula we set $y_0 = 1$. Then the model parameters to be calibrated are the corresponding function values y_i for the linearly interpolated grid: (y_1, \dots, y_5) .⁶

The calibration results for several dates in the time period from 26 April 2005 to 07 June 2005 are shown in Tables 2 to 3. Tranche market data for the year 2005 is from CreditFlux whereas that for 2007 is from DKIB Credit Research. Single name credit market data is from DKIB Credit Research. Values for the 0-3% tranche are upfront prices assuming a running spread of 500bps⁷, whereas for all other tranches the par spread is given in bps.

Table 2 shows the market and calibration results for 26 April 2005. This is before the tranche market turmoil in May 2005 and corresponds more or less to a normal market situation. All models considered here perform reasonably well, even though STABLEmix seems

⁶The capability of the model to react sensitively enough to a change of the function values y_i depends strongly on the choice of the parameters x_i . Note, that the copula function is called with $P_i(t)$ as arguments, therefore choosing “more” x_i values in a range where the P_i ’s are denser turned out to be a good choice. We found the values by manual experimentation. This procedure turned out to be superior to choosing the x_i values with the help of the solver, i.e. incorporating it into the optimization procedure. Since the curve can be arbitrarily refined, there is a lot of improvement possible by making use of more advanced parametrizations, which, however, is not the focus of the current paper.

⁷1bps = 0.01% being one *basis point*

26-Apr-05						Parameters				
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	q_a	q_b	λ	α	β
GAUSSmix	26.1%	161	50	30	18	0.367	0.183	0.603		
CAUCHYmix	26.0%	93	33	33	24	0.981	0.279	0.007		
STABLEmix	26.0%	160	47	29	17	0.080	0.128	0.582	1.806	0.203
STABLE	26.1%	159	55	31	18	0.000	0.000	0.439	1.759	-0.040
						λ	ψ	θ	ϕ	
COPmixINH	26.5%	171	72	42	20	0.717	0.098	0.100	0.010	
COPmixHOM	26.5%	169	74	43	20	0.210	0.566	0.006	0.010	
						y_1	y_2	y_3	y_4	y_5
ARCHIMpl	25.9%	155	54	26	15	0.603	0.296	0.223	0.135	0.071
Market	25.9%	156	50	24	14					

Table 1: Calibration results for 5 year iTraXX Series 3 for 26 April 2005.

to be slightly better in the upper (so called “mezzanine” and “senior”) tranches. The effect of introducing stochastic correlations under these market conditions for the α -stable copula is relatively small, as can be seen by comparing STABLE to STABLEmix.

26-Apr-05						Parameters				
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	q_a	q_b	λ	α	β
GAUSSmix	26.1%	161	50	30	18	0.367	0.183	0.603		
CAUCHYmix	26.0%	93	33	33	24	0.981	0.279	0.007		
STABLEmix	26.0%	160	47	29	17	0.080	0.128	0.582	1.806	0.203
STABLE	26.1%	159	55	31	18	0.000	0.000	0.439	1.759	-0.040
						λ	ψ	θ	ϕ	
COPmixINH	26.5%	171	72	42	20	0.717	0.098	0.100	0.010	
COPmixHOM	26.5%	169	74	43	20	0.210	0.566	0.006	0.010	
						y_1	y_2	y_3	y_4	y_5
ARCHIMpl	25.9%	155	54	26	15	0.603	0.296	0.223	0.135	0.071
Market	25.9%	156	50	24	14					

Table 2: Calibration results for 5 year iTraXX Series 3 for 26 April 2005.

Tables ?? to ?? show the market and calibration results for several dates during the market dislocation in the correlation crisis of May 2005. For these markets the Gaussian and other copula models do not perform well, whereas the α -stable based models (CAUCHYmix, STABLEmix, STABLE) capture these markets very well, as is evidenced in particular by the results of 17 May and 24 May. Note that for these markets the calibration results in α -values of around 1.3 (for CAUCHYmix it is 1 by definition), which is far away from the normal distribution value of $\alpha_{GAUSS} = 2$. The effect of stochastic correlation for the α -stable model (STABLEmix vs. STABLE) still seems relatively small in this market regime as well, with STABLEmix showing a slightly better fit.

Table 3 shows the market and calibration results for 07 June 2005. This is now after the tranche market turmoil in May 2005 and again corresponds to a return to a normal market situation with upfront prices for the 0–3% tranche having come back considerably from where

07-Jun-05						Parameters				
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	q_a	q_b	λ	α	β
GAUSSmix	31.0%	105	34	26	14	0.826	0.164	0.472		
CAUCHYmix	30.9%	81	24	22	17	0.889	0.122	0.085		
STABLEmix	31.0%	106	34	23	15	0.193	0.009	0.514	1.631	-0.046
STABLE	31.0%	106	34	24	15	0.000	0.000	0.570	1.671	-0.415
						λ	ψ	θ	ϕ	
COPmixINH	31.5%	124	57	38	18	1.478	0.865	0.007	0.010	
COPmixHOM	31.5%	126	59	39	17	0.361	0.951	0.003	0.010	
						y_1	y_2	y_3	y_4	y_5
ARCHIMpl	30.7%	116	30	19	14	0.596	0.296	0.236	0.170	0.089
Market	31.0%	106	34	24	15					

Table 3: Calibration results for 5 year iTraXX Series 3 for 07 June 2005

06-July-07					
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%
5 year					
STABLEmix	10.56%	44	18	13	9
Market	11.37%	62	17	8	3
7 year					
STABLEmix	25.59%	144	42	22	12
Market	25.36%	141	39	19	9
10 year					
STABLEmix	39.68%	374	132	60	23
Market	39.10%	365	112	52	17

Table 4: Results for different tenors when calibrated to the 7 year iTraXX Series 7 tranches.

they where in May. Except for CAUCHYmix all models considered here perform fairly well, with the α -stable models providing the best fit. As can be seen once more from comparing STABLE to STABLEmix, there is no discernible effect of introducing stochastic correlation in these markets.

The calibration results show that the stable model with stochastic correlation STABLEmix performs best throughout this time of market turmoil in the spring of 2005. But even the pure stable model STABLE (without stochastic correlation), which uses only three parameters (α , β and λ) to describe the distribution, calibrates reasonably well to these markets. In particular, these two models show mezzanine tranche and super senior spreads closer to the market than the other models. Given the market turbulence in May 2005, the deviations for the stable models from the market (i.e. the calibration error) may well be within the bid/offer spread at that time, but we were unable to confirm this. During normal markets (e.g. 07 June 2005) the stable model STABLE alone may be sufficient to calibrate to the quoted tranche spreads. The piecewise linear Archimedean copula ARCHIMpl also yields higher spreads than a simple first-generation Gaussian copula for the upper tranches, but tends to overprice the first mezzanine tranche and to remain below the market spread for the 12%-22% tranche.

As the suitability of the α -stable models for tranche modeling is the major focus of this paper we add two further outputs of the stochastic correlation version (STABLEmix) of this model. The first concerns the ability to match the market for various tenors and the second shows the tranchelet spread curve of the model.

Table 4 displays the performance of the STABLEmix model to replicate the markets for different tenors, when calibrated only to the market at one tenor. The results in this table are from 06 July 2007. The STABLEmix model was calibrated to the 7 year iTraxx tranche market and the spreads for the 5 year 10 year tenors were calculated from the model calibrated to the 7 year markets. While the fit to the 5 and 10 year markets is not too bad, it is clearly not good enough to use the model without further adjustments for a tenor structure. Such adjustments could e.g. consist of “patching” or “bootstrapping” various copula time-slices together as has recently been proposed by Sidenius [14].

Finally, the prices from STABLEmix for 1% wide 5 year tranchlets on the iTraxx series are shown in Fig. 3. The spreads show a smooth and rapid decay to zero at detachment of 60%, which is given as the maximal portfolio loss due to an assumed recovery of 40%.

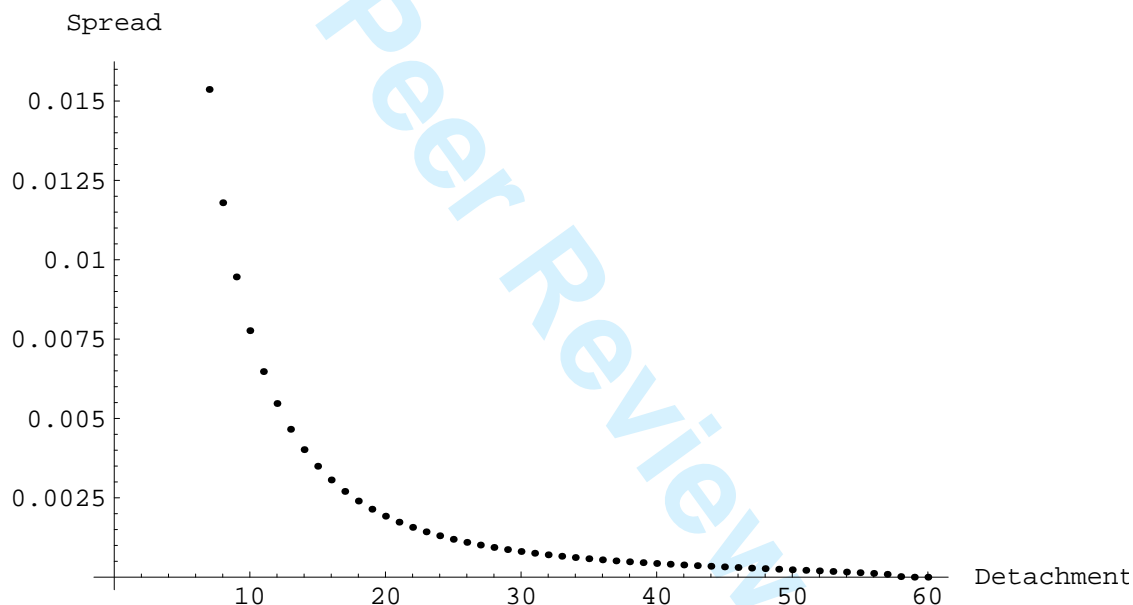


Figure 3: Tranchelet spreads from the STABLEmix model for 5 year 1% wide tranches on the iTraxx 7 as of 26 July 2007. Detachment is given in % of portfolio notional.

4 Conclusion

We have introduced a two-parameter (α, β) class of distributions closed under addition, so-called α -stable distributions, which include the standard Gaussian distribution as a special case ($\alpha = 2, \beta = 0$), but allow a continuous deformation of the latter to distributions with ever fatter tails as α decreases away from 2 and more skewness as β is changed away from 0. Using these α -stable distributions for the idiosyncratic and latent variables in conditional independence loss models results in significantly higher tranche spreads for the upper tranches than those which can be achieved with the standard Gaussian models. Combining the α -stable

distribution with the framework of stochastic correlation further enhances the ability to match the market and yields quite good calibration properties for these models, as evidenced by applying them to the markets during the correlation crisis of 2005.

Furthermore, we have carried out the search for a factor dependence structure given by a “perfect” Archimedean copula with a generator which is allowed to be piecewise linear and whose interpolation points are determined by calibration. The resulting class of models also allow for higher spreads in the upper tranches and a correlation smile, but at the same time tend to deliver too high spreads for the first mezzanine tranches.

Overall we believe that the α -stable mixture models with stochastic correlation comprise a class of models capable of capturing the tranche market.

References

- [1] David X. Li. On default correlation: A copula function approach. *The Journal of Fixed Income*, 9(4):43–54, March 2000.
- [2] Leif Andersen and Jakob Sidenius. Extension to the gaussian copula: Random recovery and random factor loadings. *Journal of CreditRisk*, 1(1):29–70, March 2004.
- [3] X. Burtschell, J. Gregory, and J-P. Laurant. Beyond the gaussian copula: Stochastic and local correlation. http://laurent.jeanpaul.free.fr/stochastic_local_correlation.pdf, 2005.
- [4] A. Kalemánova, B. Schmidt, and R. Werner. The normal inverse gaussian distribution for synthetic CDO pricing. <http://www.mathfinance.ma.tum.de/personen/KalemánovaSchmidWerner.pdf>, 2005.
- [5] X. Burtschell, J. Gregory, and J-P. Laurant. A comparative analysis of CDO pricing models. <http://www.mathematik.uni-ulm.de/finmath/ss.05/fe/jplaurantcomparative.pdf>, 2005.
- [6] Philipp J. Schönbucher and Ebbe Rogge. Modeling dynamic portfolio credit risk. www.finasto.uni-bonn.de/~schonbuc/papers/schonbucher_rogge_dynamiccreditrisk.pdf, 2003.
- [7] J. Hull and A. White. The perfect copula. <http://www.rotman.utoronto.ca/~hull/DownloadablePublications/PerfectCopula.pdf>, 2005.
- [8] Hansjörg Albrecher, Sophie Ladoucette, and Wim Schoutens. A generic one-factor levy model for pricing synthetic cdos. <http://perswww.kuleuven.be/~u0009713/LevyCDO.pdf>, 2006.
- [9] Thomas Moosbrucker. Pricing cdos with correlated variance gamma distributions. <http://www.fmpm.ch/docs/9th/papers.2006-web/9112b.pdf>, 2006.
- [10] Martin Baxter. Dynamic modelling of single-name credits and cdo tranches. <http://www.nomura.com/resources/europe/pdfs/dynamiccdomodelling2.pdf>, 2006.

- 1
2
3
4
5
6
7 [11] P. Tavares, Th.-U. Nguyen, A. Chapovsky, and I. Vaysburd. Composite basket model.
8 Technical report, Merrill Lynch Credit Derivatives, July 2004.
9
10 [12] John P. Nolan. *Stable Distributions*. in preparation, 2005.
11
12 [13] Roger B. Nelsen. *An Introduction to Copulas*. Number 139 in Lecture Notes in
13 Statistics. Springer, 1999.
14
15 [14] Jakob Sidenius. On the term structure of loss distributions – a forward model
16 approach. <http://www.jakobsidenius.com/papers/forwardcopula.pdf>, 2006.
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

- $\alpha = 1.5, \beta = 0$
- $\alpha = 1.7, \beta = 0$
- $\alpha = 1.3, \beta = 0$

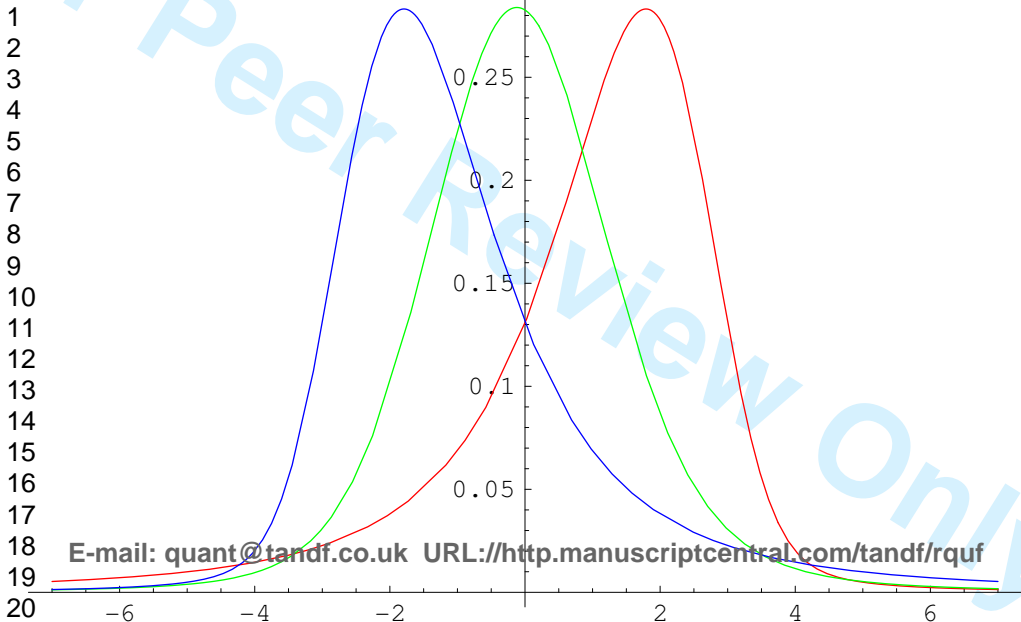
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26

0.3
0.25
0.2
0.15
0.1
0.05

-6 -4 -2 2 4 6

E-mail: quant@tandf.co.uk URL: <http://manuscriptcentral.com/tandf/rquf>

alpha = 1.3, beta = -0.8
alpha = 1.7, beta = 0.2
alpha = 1.3, beta = 0.8



E-mail: quant@tandf.co.uk URL: <http://manuscriptcentral.com/tandf/rquf>

Spread

