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Bond Pricing when the Short-Term Interest Rate Follows a Threshold Process *

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Abstract

This paper derives analytical solutions for arbitrage-free bond yields when the short-term interest rate follows an autoregressive process with the intercept switching endogenously. This process from the SETAR family is especially suited to capture the near-unit-root behavior typically observed in the evolution of short-term interest rates. The derived yield functions, mapping the one-month rate into n-period yields, exhibit a convex/concave shape to the left and the right of the threshold value, respectively; a pattern which is also found in US bond yield data. The longer the time to maturity, the more distinct the nonlinearity of the yield function becomes.

JEL Classification: E43, G12, C63

Keywords: Threshold process, interest rate modeling, non-affine term structure model

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1 Introduction

Within the literature on arbitrage-free term structure models, the affine class expounded by Duffie and Kan (1996) has become very popular. For models of this family, bond yields are affine functions of the driving state variables. This property follows from a linear state process and an adequately chosen pricing kernel. For the sub-family of one-factor models, the short-term interest rate follows a linear process and all long-term bond yields are affine functions of the short rate.[1] While this is a convenient property of linear models, the empirical literature on interest rate dynamics finds evidence for nonlinearities in short-rate dynamics. This leads to the question of how certain forms of nonlinear dynamics translate into the cross-sectional relationship between bond yields of different maturities.

This paper addresses this issue for a discrete-time threshold process governing the evolution of the short rate. This econometric specification is presented by Lanne and Saikkonen (2002) and is especially suited to capture the near unit-root dynamics of interest rates. We consider the simplest version of their model in which the law of motion is a first-order autoregressive process with homoscedastic Gaussian innovations. The intercept is allowed to switch between two regimes. The regime prevailing is determined by the previous period's realization of the short rate, i.e. the model is of the SETAR (Self Exciting Threshold Autoregression) type.

The key contribution of this paper is the derivation of an analytical solution for arbitrage-free long-term bond yields for the case that the short-term interest rate follows the SETAR process. Compared to an affine Gaussian one-factor model, the only difference of our state process is the changing intercept. However, it turns out that this slight modification induces substantial changes to the solution compared to the affine model. The yield function, mapping realizations of the short rate into yields for longer maturities, is nonlinear and exhibits a point of discontinuity at the threshold value. The function is convex to the left and concave to the right of the threshold value. This convex-concave shape matches similar patterns observed in the data. For values of the short rate sufficiently far off the threshold value, however, the yield function is approximately linear. This is in sharp contrast to one-factor models from the affine family, for which the yield function is linear for any level of the short rate. The approach that we employ to derive arbitrage-free bond yields is transferable to the case of more

[1] The properties of affine models are analyzed by Dai and Singleton (2000). The discrete-time version of the affine class is described by Backus, Foresi, and Telmer (1998). The one-factor models of Vasiček (1977) and Cox, Ingersoll, and Ross (1985) constitute the most prominent examples from this class.

elaborate threshold processes of the short rate.

Our study relates to other work in the literature that derives the term structure of interest rates based on regime-switching state processes in discrete time. One part of this literature is concerned with models, in which regime-shifts are governed by a hidden Markov chain process. Earlier approaches such as Hamilton (1988) and Kugler (1996) derive long-term bond yields under the expectations hypothesis. Bansal and Zhou (2002) and Bansal, Tauchen, and Zhou (2004) derive arbitrage-free bonds using a stochastic-discount-factor approach. They provide closed-form solutions for bond prices, but these are still approximate as they rely on a linearization. Under the assumption that intercepts and innovation variances of the state process are regime-dependent but persistence parameters are not, Ang, Bekaert, and Wei (2006) and Dai, Singleton, and Yang (forthcoming) derive analytical solutions for arbitrage-free zero yields. Compared to models from the affine class, the intercept in the relation between long-term yields and factors is now regime-dependent, while the factor loadings are constant across regimes. On the one hand, this is parallel to the structure of our solution, where the intercept depends on the previous period's level of the short rate, but the loading of the short rate on bond yields is regime-independent. On the other hand, in contrast to the Markov-switching case, the prevailing regime – and thus the intercept in the pricing equation – is directly measurable from observed data.

Threshold models, in which the prevailing regime depends on a predetermined observable state variable, are used by Pfann, Schotman, and Tschernig (1996) and Gospodinov (2005). They employ SETAR and TAR-GARCH specifications, respectively, which allow for more flexible dynamics as our model.[2] As such, however, arbitrage-free bond yields can only be computed via simulation methods. The model by Audrino and Giorgi (2005) comprises discrete beta-distributed regime shifts constructed on multiple thresholds. The *expected* regime for a period depends on the previous period's state, but the regime is not fully predetermined as in our case. Their model allows for an iterative closed-form formula for arbitrage-free yields. Finally, Decamps, Goovaerts, and Schoutens (2007) can be interpreted as the continuous-time analogue to our study. The authors obtain closed-form solutions for state price densities for self-exciting-threshold variants of the continuous-time Vasiček and CIR models.

The structure of the paper is as follows. The next section gives a description of the model, section 3 derives the analytical yield function, followed by a discussion of numerical problems and an empirical illustration in section 4. The fifth section concludes, an appendix contains details of the derivation of the yield function.

[2]In both papers, the intercept, the persistence parameter and the volatility is allowed to switch. Gospodinov (2005) also superimposes a GARCH specification.

2 Modeling the Short Rate as a Self-Exciting Threshold Process

The model for the dynamics of the short rate belongs to the class of self-exciting threshold autoregressive (SETAR) processes treated by Lanne and Saikkonen (2002). Time is discrete, and one period may correspond to one month as in the empirical study by Lanne and Saikkonen. The real-valued one-month rate X_t follows

$$X_t = \nu + \sum_{k=1}^{r} \beta_k \, I(X_{t-d} \geq c_k) + \sum_{j=1}^{p} \kappa_j X_{t-j} + \sigma(X_{t-d})\epsilon_t. \tag{2.1}$$

where

$$\sigma(X_{t-d}) := \sigma + \sum_{k=1}^{r} \omega_k I(X_{t-d} \geq c_k),$$

$I(\cdot)$ is the indicator function and ϵ_t is a serially-independent innovation with $E\epsilon_t = 0$ and $E\epsilon_t^2 = 1$, and ν, β_k, c_k, κ_j, and ω_k are parameters. The roots of the polynomial $\kappa(L) = 1 - \sum_{j=1}^{p} \kappa_j L^j$ are assumed to be outside the unit circle to guarantee geometric ergodicity for X_t. The model can be referred to as a SETAR(r, p, d), where r indicates the number of level shifts, p the lag order, and d the lag of the threshold variable. When all β_k and all ω_k are zero, the model reduces to the standard linear and homoscedastic AR(p) model. Otherwise, the intercept and/or the variance are regime-dependent. There are $r+1$ different regimes, where the regime prevailing in period t depends on the value of the short rate at time $t-d$.

The above specification allows for endogenous regime shifts of the short-term interest rate's level and its volatility. The switching-intercepts specification allows the short rate process to revert to different means. This is important for econometric analyses as ignoring these regime shifts would lead to an overestimation of the process's persistence, potentially leading to a false decision in favor of a unit-root process.[3] Moreover, the variance specification can account for the stylized facts that episodes of high interest rates are associated with higher volatility.

Just like the linear (discrete-time) Vasiček model, the SETAR specification (2.1) implies that the short rate can become negative. In a continuous-time framework, it is well known that the Cox et al. (1985) specification implies a zero probability of negative interest rates. In the discrete-time context considered here, one may similarly specify the innovation as proportional to the square root of the lagged short rate. If the drift of the short rate were a linear autoregressive process this would still imply a strictly positive probability of negative short rates, but this probability would decrease as the time interval becomes smaller.

[3]See Lanne and Saikkonen (2002).

However, as the starting point of this paper is a specific class of short-rate specifications from the econometrics literature, we do not want to consider models outside class defined by (2.1). But even the formulation of the innovation variance in these models can be interpreted as a discretized variant of the CIR approach. By choosing positive ω_k parameters in (2.1), the innovation variance decreases with the level of the short rate but can only assume $r+1$ different values instead of changing continuously. Simulation studies show that this variance specification will also reduce the probability of negative short rates compared to the homoscedastic case.

As SETAR specifications prove empirically successful in matching the dynamic properties of the short rate, the question arises how the endogenous regime-shifts translate to long-term bond yields. More specifically, it is worthwhile exploring how the respective bond price equations differ from those associated with an affine short-rate process, which is nested in (2.1) as a special case. In the following we will derive arbitrage-free bond yields for the case that short rate dynamics are governed by the simplest non-trivial member of the family of processes defined by (2.1). That is, in the remainder of the paper we consider the SETAR(1,1,1)

$$X_t = \nu + \beta\, I(X_{t-1} \geq c) + \kappa\, X_{t-1} + \sigma\epsilon_t, \quad \epsilon_t \sim\ N(0,1). \tag{2.2}$$

The only difference to a linear Gaussian model – i.e. the discrete-time version of the Vasiček model[4] – is the time-varying intercept. Depending on the previous realization of the short rate it is given by ν or $\nu+\beta$, respectively. By sticking to the simple SETAR(1,1,1), as opposed to the general (2.1), we can best accentuate the effects on the term structure of interest rates that are implied by only this slight modification of the purely linear case. Moreover, the solution approach that we take should be generalizable to more general cases, but we think that its structure can be made most transparent when concentrating on the special case.

Let P_t^n denote the time t price of a default-free zero-coupon bond with n periods left until maturity. The payoff is normalized to one, so $P_t^0 = 1$. Continuously compounded monthly yields are computed from bond prices as

$$y_t^n = -\frac{\ln P_t^n}{n}. \tag{2.3}$$

Absence of arbitrage is equivalent to the existence of a strictly positive stochastic discount factor (SDF) process $\{M_t\}$, with $E|M_t P_t^n| < \infty$ and

$$P_t^n = E(M_{t+1} P_{t+1}^{n-1} | \mathcal{F}_t), \tag{2.4}$$

[4] See Backus et al. (1998).

for all n and t, where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{X_{t-i}\}_{i\geq 0}$. Since the short-rate dynamics has the Markov property, any expectation over the future conditional on $\mathcal{F}_t$ equals the expectation conditional on the information contained in X_t alone. We will thus write the basic pricing equation simply as

$$P_t^n = E(M_{t+1} P_{t+1}^{n-1} | X_t). \tag{2.5}$$

For the stochastic discount factor we assume

$$M_{t+1} = \exp\{-\delta - X_t - \lambda\sigma\epsilon_{t+1}\}, \quad \delta = \frac{1}{2}\sigma^2\lambda^2 \tag{2.6}$$

where the exponential specification is chosen to guarantee positivity. The parameter λ is referred to as the market price of risk, it governs the covariance of shocks to the state variable and the discount factor.[5]

The model specification is now complete: given the state process (2.2) and the pricing kernel specification (2.6), arbitrage free bond price processes $\{P_t^n\}$ are given as the solution of the stochastic difference equation (2.5). An explicit solution of the model writes bond prices, or equivalently yields, as functions of the factor X_t, i.e they are of the form

$$y_t^n = f_n(X_t; \psi), \tag{2.7}$$

where ψ collects all model parameters. The next section is devoted to finding this solution function f_n for our threshold model.

3 Arbitrage-free Term Structure

We start by writing bond prices as a function of future discount factors. Substituting the basic pricing equation (2.5) repeatedly into itself, using the law of iterated expectations and noting that $P_t^0 = 1$, we can write the time t price of the n-period bond as

$$P_t^n = E(M_{t+1} \cdot M_{t+2} \cdot \ldots \cdot M_{t+n} | X_t), \tag{3.1}$$

equivalently using discount factors in logs,

$$P_t^n \quad = \quad E(\exp[m_{t+1} + \ldots + m_{t+n}] | X_t). \tag{3.2}$$

Before we turn to the model based on (2.2) it is instructive to consider the special case of $\beta = 0$, that is with X_t following the linear Gaussian process

$$X_t = \nu + \kappa X_{t-1} + \sigma\epsilon_t, \quad \epsilon_t \sim N(0,1). \tag{3.3}$$

[5] See Backus et al. (1998) and Cochrane (2001).

Since for this case X_t is a linear process, the sum of log SDFs can be written as a linear combination of X_t and future ϵ_t only. This yields for the bond price

$$P_t^n = E\left(\exp\left[-a_n - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i}\right]\middle| X_t\right) \tag{3.4}$$

where a_n, B_n and b_n^i are coefficients depending on the model parameters ν, κ, σ, and λ as well as on time to maturity n. The important point is the exponential-affine structure. The exact form of these coefficients, expressed in terms of ν, κ, σ, and λ is not relevant here. For the threshold model (which nests the linear model) they are given in proposition 3.1 below.

Since the ϵ_t are Gaussian white noise, their sum is also normal and the exponential expression in (3.4) has a conditional lognormal distribution. Computing the required expectation yields the solution

$$P_t^n = \exp[-A_n - B_n X_t], \tag{3.5}$$

where

$$B_n = \sum_{i=0}^{n-1} \kappa^i = \frac{1-\kappa^n}{1-\kappa}, \tag{3.6}$$

$$A_n = \sum_{i=0}^{n-1} G(B_i), \tag{3.7}$$

with

$$G(B_i) = \delta + B_i\,\nu - \frac{1}{2}(\lambda + B_i)^2\sigma^2.$$

Using (2.3), we obtain bond yields as an affine function of the short-term interest rate,

$$y_t^n = \frac{A_n}{n} + \frac{B_n}{n} X_t. \tag{3.8}$$

Note that this implies that for a given time to maturity n, the sensitivity of yields with respect to interest rate changes does not depend on the level of the short rate.

Usually, bond prices for the linear Gaussian case are obtained using a method of undetermined coefficients.[6] One assumes that bond prices are in fact of the form (3.5) and inserts this expression on both sides of (2.5). It turns out that for A_n and B_n (viewed as a function of n) to satisfy (2.5) for all n and t, they have to solve a system of difference equations the solution of which is given by (3.6) and (3.7). Here, in contrast, we have chosen to show the solution based on directly computing the expected product of future stochastic discount factors,

[6] Cf. Backus et al. (1998) or Cochrane (2001).

(3.1). This is because we will take the same route now when deriving bond prices for the threshold case, which would not allow using the method of undetermined coefficients.

For the case that the short rate follows the threshold process (2.2), the product of pricing kernels in (3.1) does not have a conditional lognormal distribution anymore. The threshold specification implies that future pricing kernels will inter alia depend on the path of future intercepts. This sequence is in turn a nonlinear function of the path of the short rate. Thus, the endogenously switching intercept implies that bond prices are no longer exponentially-affine functions of future innovations. It turns out this makes the computation of bond prices a much more intricate task compared to the class of affine models. The key idea to handle the dependence on future intercepts when computing arbitrage-free bonds is as follows: we first compute the conditional expectation in (3.2) conditional on a particular sequence of future intercepts and then take a probability-weighted average over all such sequences. The following will describe the basic idea of that approach and state the exact solution. The details of the derivation are delegated to the appendix.

For $n = 1$, equation (3.2) should lead to the short rate itself, i.e. $y_t^1 = X_t$. This is in fact the case since

$$P_t^1 = E(M_{t+1} \cdot 1|X_t) = E(\exp[-\delta - X_t - \lambda\sigma\epsilon_{t+1}])|X_t) = \exp[-\delta - X_t + \lambda^2\sigma^2],$$

and thus,

$$y_t^1 = \delta + X_t - \frac{1}{2}\lambda^2\sigma^2 = X_t.$$

For treating maturities $n > 2$ we introduce the notation

$$S_t = I(X_t \geq c) \text{ and } a(S_t) = \nu + \beta S_t,$$

and will refer to the binary random variable S_t as the regime of the system at time t. Thus, for the threshold process (2.2),

$$X_{t+1} = a(S_t) + \kappa X_t + \sigma\epsilon_{t+1}. \tag{3.9}$$

The price of the two-period bond is given by

$$\begin{aligned} P_t^2 &= E(\exp[m_{t+1} + m_{t+2}]|X_t) \\ &= E\left(\exp\left[-2\delta - X_t - X_{t+1} - \sigma\lambda(\epsilon_{t+1} + \epsilon_{t+2})\right]|X_t\right) \\ &= \exp\left[-2\delta - X_t - a(S_t) - \kappa X_t\right] \cdot E\left(\exp\left[-\sigma(1+\lambda)\epsilon_{t+1} - \sigma\lambda\epsilon_{t+2}\right]|X_t\right) \end{aligned}$$

Conditional on X_t, the last exponent is normally distributed with mean 0 and variance $\sigma^2((1+\lambda)^2 + \lambda^2)$. Thus, the exponential expression has a conditional lognormal distribution, and

$$E\left(\exp\left[-\sigma(1+\lambda)\epsilon_{t+1}-\sigma\lambda\epsilon_{t+2}\right]|X_t\right)=\exp\left[\frac{1}{2}\sigma^2((1+\lambda)^2+\lambda^2)\right].$$

Collecting terms delivers

$$P_t^2=\exp[-A_2(X_t)-B_2X_t] \tag{3.10}$$

with

$$A_2(X_t)=a(S_t)+2\delta-\frac{1}{2}\sigma^2(\lambda^2+(1+\lambda)^2) \tag{3.11}$$

and

$$B_2=(1+\kappa). \tag{3.12}$$

Hence, using (2.3), for the yield we obtain

$$y_t^2=\frac{A_2(X_t)}{2}+\frac{B_2}{2}X_t. \tag{3.13}$$

The derivation has employed the same techniques as in the purely Gaussian case. The structure of the solution, however, does differ from (3.8). The two-period yield is a stepwise linear function of the short rate: the intercept depends on $a(S_t)\equiv\nu+\beta\cdot I(X_t\geq c)$. Thus, y_t^2 viewed as a function of X_t features a discontinuity at $X_t=c$. However, at all points of continuity, the derivative of the two-month yield with respect to the short rate is constant. Moreover, the expression B_n is the same as in the linear case.

For $n>2$ the solution of the bond price can be written in a similar form as in (3.4). However, since the underlying short-rate process now involves the time-varying intercepts, the representation of future log SDFs involves not only future ϵ_t but also future intercepts which in turn are dependent on future X_t. In the appendix it is shown that bond prices can be written as

$$\begin{aligned}P_t^n &= E(\exp[m_{t+1}+\ldots+m_{t+n}]|X_t)\\ &= E\left(\exp\left[-n\delta-B_nX_t+\sum_{i=1}^{n}b_i^n\epsilon_{t+i}+\sum_{j=0}^{n-2}c_j^n\cdot a(S_{t+j})\right]\middle|X_t\right),\end{aligned}$$

where B_n, b_i^n, c_j^n are coefficients depending on the model parameters κ, σ and λ. The crucial thing to note is that the expression in parentheses is not a linear function of future X_t anymore as it was in the case of a simple linear AR(1) for X_t.[7] Accordingly, conditional on X_t, the expression is not lognormal. Our solution for this case makes use of a similar idea as employed in Bansal and Zhou (2002). We will evaluate the expression by first computing the expectation

[7]Recall that $a(S_t)=\nu+\beta\cdot I(X_t\geq c)$.

for an arbitrary *given* realization of the regime sequence $(S_{t+1}, \ldots, S_{t+n-2})'$, say $(\bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2})'$, and then take the probability-weighted sum over all possible realizations of $(S_{t+1}, \ldots, S_{t+n-2})'$. That is, we first enlarge the conditioning information set and then integrate out the enlargement again. As S_{t+i} can assume two different values, 1 and 0, the number of different extended information sets, $\{X_t, \bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2}\}$, amounts to 2^{n-2}. It is obvious that this will be one obstacle for obtaining numerical values for bond yields with longer times to maturity.

Under the extended information set $\{X_t, \bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2}\}$, the exponential expression above does not have a plain lognormal distribution. This is because $(S_{t+1}, \ldots, S_{t+n-2})'$ and $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ are not independent. In other words, knowing that a particular path of intercepts $(a(\bar{S}_{t+1}), \ldots, a(\bar{S}_{t+n-2}))'$ has been realized, restricts the set of possible realizations of $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$: conditional on the extended information set, $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ has a *truncated* multivariate lognormal distribution.

The following proposition states the solution for bond yields with time to maturity exceeding two months.

Proposition 3.1 (Yield function for $n > 2$) *For the short rate process given by (2.2) and the pricing kernel defined by (2.6), yields with time to maturity $n > 2$ as a function of the short rate X_t are given as:*

$$y_t^n = \frac{A_n(X_t)}{n} + \frac{B_n}{n} X_t \tag{3.14}$$

with

$$B_n = \frac{1 - \kappa^n}{1 - \kappa} \tag{3.15}$$

and

$$\begin{aligned} & A_n(X_t) \\ = \; & n \cdot \delta - c_0^n a(S_t) - \frac{1}{2} b' b \\ & - \ln \left(\sum_{k=1}^{2^{n-2}} F\left(\tilde{h}(k);\, \tilde{H}(k) b^*,\, \tilde{H}(k)\tilde{H}(k)'\right) \exp\left[\sum_{j=1}^{n-2} c_j^n \cdot a\left(\bar{S}_{t+j}(k)\right) \right] \right) \end{aligned} \tag{3.16}$$

which uses the following definitions:

$$b = (b_1^n, \ldots, b_n^n)', \quad b^* = (b_1^n, \ldots, b_{n-2}^n)' \text{ where } b_i^n = -\sigma \left(\lambda + \frac{1 - \kappa^{n-i}}{1 - \kappa} \right)$$

$$c_j^n = -\frac{1 - \kappa^{n-j-1}}{1 - \kappa}, \quad j = 0, 1, \ldots, n-2$$

The function $F(r; \mu, \Sigma)$ denotes the cumulative distribution function of the multivariate normal $N(\mu, \Sigma)$ evaluated at the vector r.

The first summation in (3.16) runs over all possible realizations of the sequence $\{S_{t+1}, \ldots, S_{t+n-2}\}$, i.e. over all possible sequences of length $n-2$ that consist of zeros and ones. $\{\bar{S}_{t+1}(k), \ldots, \bar{S}_{t+n-2}(k)\}$ denotes a particular sequence of this sort. The indexing may be such that k is the decimal number (plus one) that corresponds to the binary number represented by the sequence. For instance, the sequence

$$\{\bar{S}_{t+1}(k), \bar{S}_{t+2}(k), \bar{S}_{t+3}(k), \bar{S}_{t+4}(k)\} = \{1, 0, 0, 1\}$$

corresponds to the decimal number 9 and would carry the index $k = 10(= 9 + 1)$.

The vector $\tilde{h}(k)$ is given by[8]

$$\tilde{h}(k) = \tilde{c}(k) - \tilde{f}(k) \cdot X_t - \tilde{G}(k) \cdot a\big(\bar{\zeta}_t^*(k)\big)\,. \tag{3.17}$$

The remaining expressions are defined as follows:

$$\bar{\zeta}_t^*(k) = (S_t, \bar{S}_{t+1}(k), \bar{S}_{t+2}(k), \ldots, \bar{S}_{t+n-3}(k))', \tag{3.18}$$

$$a(\bar{\zeta}_t^*(k)) = (a(S_t), a(\bar{S}_{t+1}(k)), a(\bar{S}_{t+2}(k)), \ldots, a(\bar{S}_{t+n-3}(k)))', \tag{3.19}$$

$$\tilde{f}(k) = \begin{pmatrix} R(\bar{S}_{t+1}(k)) \cdot \kappa^1 \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \cdot \kappa^{n-2} \end{pmatrix} \tag{3.20}$$

where

$$R(\bar{S}) = \begin{cases} 1, & \textit{if } \bar{S} = 0 \\ -1, & \textit{if } \bar{S} = 1, \end{cases} \tag{3.21}$$

$$\tilde{G}(k) = \begin{pmatrix} \tilde{g}_1^1(k) & 0 & 0 & \ldots & 0 \\ \tilde{g}_1^2(k) & \tilde{g}_2^2(k) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{g}_1^{n-2}(k) & \tilde{g}_2^{n-2}(k) & \tilde{g}_3^{n-2}(k) & \ldots & \tilde{g}_{n-2}^{n-2}(k) \end{pmatrix} \tag{3.22}$$

with

$$\tilde{g}_j^i(k) = R(\bar{S}_{t+j}(k)) \cdot \kappa^{i-j}, \tag{3.23}$$

$$\tilde{c}(k) = c \cdot \begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix}, \tag{3.24}$$

[8]The expressions $\tilde{h}(k)$, $\tilde{c}(k)$, $\tilde{f}(k)$, $\tilde{G}(k)$, $\tilde{c}(k)$, $\tilde{H}(k)$ depend on n and t, but we omit to indicate this explicitly.

and

$$\tilde{H}(k) = \begin{pmatrix} \tilde{h}_1^1(k) & 0 & 0 & \dots & 0 \\ \tilde{h}_1^2(k) & \tilde{h}_2^2(k) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{h}_1^{n-2}(k) & \tilde{h}_2^{n-2}(k) & \tilde{h}_3^{n-2}(k) & \dots & \tilde{h}_{n-2}^{n-2}(k) \end{pmatrix} \tag{3.25}$$

with

$$\tilde{h}_j^i(k) = R(\bar{S}_{t+j}(k)) \cdot \sigma\kappa^{i-j}. \tag{3.26}$$

Given the parameters of the threshold model, ν, δ, c, κ, σ and λ, the formula stated in proposition 3.1 allows to compute the whole arbitrage-free term structure, i.e. any n-period yield that corresponds to a realization X_t of the short rate.

4 A Quantitative Illustration

4.1 Numerical Problems

The previous section provided an analytical solution for bond yields. However, for larger n, the computation of the exact yield function runs into the following numerical problems. The first obstacle is, that the number of different intercept combinations increases exponentially with time to maturity: for an n-period yield, the first sum in (3.16) has to be taken over the 2^{n-2} different possible paths $\{\bar{S}_{t+1}(k), \dots, \bar{S}_{t+n-2}(k)\}$ of the regime variable. Hence, the computational burden increases exponentially with time to maturity.

Secondly, the formula involves $F\left(\tilde{h}(k);\, \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\right)$, the c.d.f. of a multivariate normal with general (i.e. non-diagonal) covariance matrix. Numerical software usually has difficulties to compute the corresponding multiple integral for higher dimensions, say exceeding 7. With GAUSS 6.0, or instance, computing the c.d.f. via the cdfmvn($\cdot$)-function for a multivariate normal with dimension 7 is quick, dimension 8 takes significantly longer, and dimension 9 leads to a breakdown. Since computing bond yields of maturity n requires the computation of a c.d.f of an $(n-2)$-variate normal, maturities exceeding nine months cannot be obtained in a straightforward fashion. This problem may be solved to some extent by using numerical techniques. For example, the Geweke-Hajivassiliou-Keane (GHK) simulator from the discrete choice literature can be adopted.[9]

[9] Comparing numerous probit simulators, Hajivassiliou, McFadden, and Ruud (1996) found the GHK simulator to be the most accurate in these settings. The GHK simulator can be downloaded from the web site econ.lse.ac.uk/∼vassilis.

Since due to exponentially increasing computing time, the exact solution is not directly implementable for maturities exceeding one year, bond prices for higher n may be computed using a simulation-based approach: conditional on a realization X_t of the short rate, one has to generate N realizations – say N=100,000 – of $(X_{t+1}, \ldots, X_{t+n-1})'$ and $(\epsilon_t, \ldots, \epsilon_{t+n})'$ and compute the corresponding function of the pricing kernel, $\exp[m_{t+1} + \ldots m_{t+n}]$. The average of the latter expression over all runs is an estimate of P_t^n in (3.2). It is also conceivable to refine such a Monte Carlo approach by combining it with elements of the analytical solution provided here.

The existence of the analytical solution enables us to explore how many Monte Carlo replications are needed to compute bond yields with a desired precision. Using the parametrization of the empirical example in the next section we computed bond yields of different maturities (up to $n = 8$) for different levels of the short rate X_t. We used the analytical formula above and compared it to the outcomes from the Monte Carlo approach with up to 500,000 replications. The results can be roughly summarized as follows: for $N = 100,000$ Monte Carlo replications, simulated yields scattered around the true value with a standard deviation of 3 basis points, the difference between the highest and the lowest computed yield amounted to about 16 basis points.[10] Using 500,000 replications reduced the standard deviation to about 1.5 and the range to 8 basis points. These margins of uncertainty turned out to be similar across different maturities and different levels of the short rate.

4.2 An Application to US Bond Yields

Before we proceed with a numerical example that shows the functional form of the yield function in the threshold model, we present the empirical counterpart of this relationship using continuously compounded zero-coupon US government bond yields from January 1960 to December 2002.[11] All bond yields are smoothed Fama-Bliss data except of the ten-year yield which is taken from FRED (Federal Reserve Economics Data).

Figure 1 displays different n-month yields plotted against the one-month short rate. A penalized spline (P-spline) is fitted through the data. In P-spline smoothing the unknown functional form is approximated by a high dimensional basis, which is then fitted to the data imposing a penalty against overfitting.[12] This

[10] To characterize these distribution we computed each yield (fixed n, fixed X_t, fixed N)) 500 times using the Monte Carlo approach.

[11] We would like to thank Monika Piazzesi for making this data set (1 month to 60 months) available on her website.

[12] See, e.g., Eilers and Marx (1996).

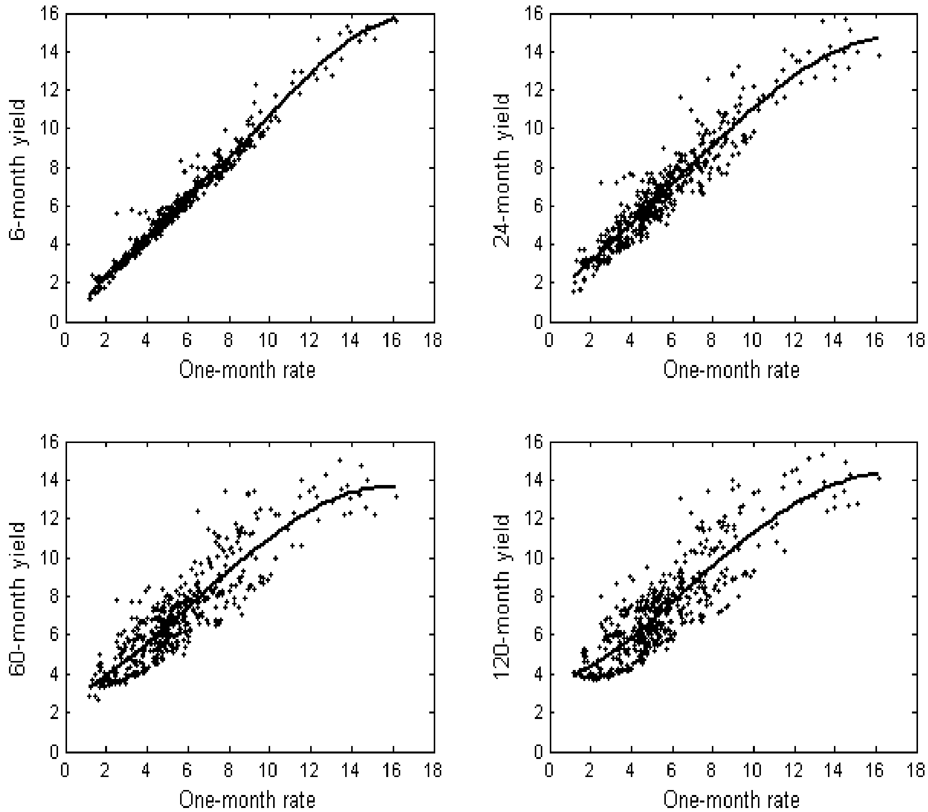

Figure 1: Scatter plots and fit between different n-month yields and the short rate.

guarantees a smooth fit while retaining the basic form of the underlying structural relationship. The figures reveal a nonlinear dependence between long-term bonds and the short rate that becomes more accentuated as time to maturity rises. The spline takes a slight convex shape when the short rate is low and exhibits concavity at higher levels of the short rate.[13] This convex/concave pattern seems to amplify for longer times to maturity. Moreover, the concavity is more pronounced indicating that long-term bonds are less sensitive to the short rate at higher levels.

We estimate the parameters of the threshold process (2.2) via the method of conditional least squares[14] using US data on the one-month interest rate. The market-price-of-risk parameter λ is calibrated such that the average ten-year yield

[13]A cubic polynomial indicates the same behavior in an even more pronounced way. Here we choose the spline to let the procedure choose a functional form that is not predetermined by itself.

[14]See Lanne and Saikkonen (2002).

in the data matched the average of model implied ten-year yields.

Based on the estimates, figure 2 draws yields of two-, three-, six-, and twelve-month yields as a function of the one-month rate. The GHK simulator is utilized to compute the twelve-month yield. The parameters are given as

$$\nu = 0.3058/1200, \quad \beta = 0.2603/1200, \quad \kappa = 0.9253,$$
$$c = 5.5296/1200, \quad \sigma = 0.7136/1200, \quad \lambda = -155.$$

Recall that for a linear one-factor model, the function that maps the short rate into n-period yields is given by (3.8), i.e. it is affine. For the threshold model,

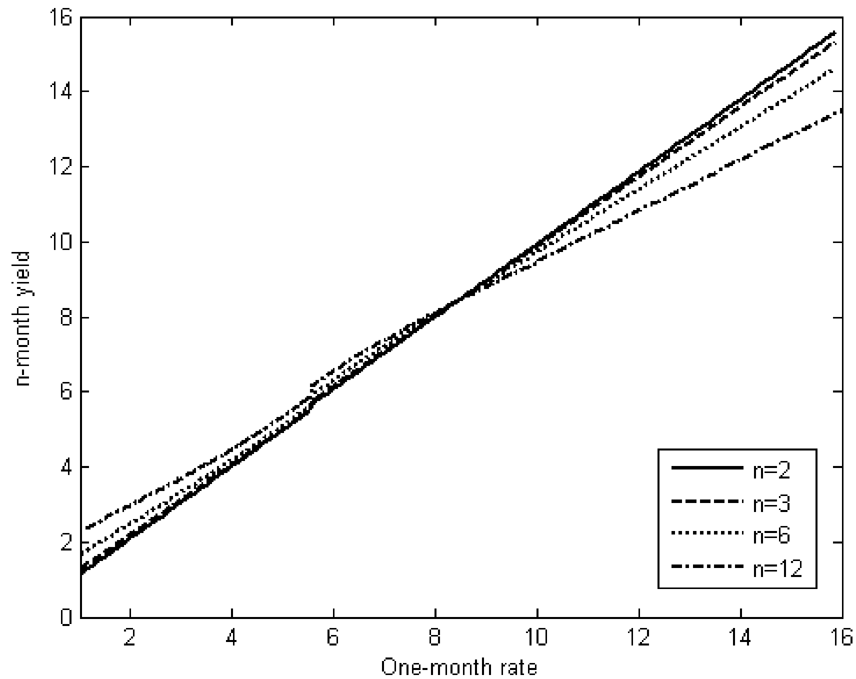

Figure 2: Different n-month yields as a function of the short rate.

the two-period yield is obtained via (3.13), a stepwise linear function, for $n \geq 3$, the yield function (given in proposition 3.1) is nonlinear. It turns out that the 'degree of nonlinearity' increases with time to maturity. However, for small n, for which yields can be actually computed, nonlinearity is hardly visible from the graph except in the case of the twelve-month yield, where a convex-concave shape to the left and the right of the threshold value is indicated.

In order to better visualize the characteristic pattern of the yield function for short maturities (for which our analytical solution is numerically implementable), we plot the second derivative of the yield function against the short rate. Let $f_n(x)$ denote the function that assigns the corresponding n-period yield to the short rate x, i.e. $f_n(x) = A_n(x)/n + B_n/n \cdot x$, with $A_n(\cdot)$ and B_n given by (3.16) and (3.15). For a small number h, we approximate the second derivative as

$$\frac{d^2 f_n(x)}{d\,x^2} \approx \frac{f_n(x-h) - 2f_n(x) + f_n(x+h)}{h^2} =: k_n(x), \tag{4.1}$$

at all points of continuity.[15] Figure 3 plots $k_n(x)$ against x for $n = 2, 3$ and 6.

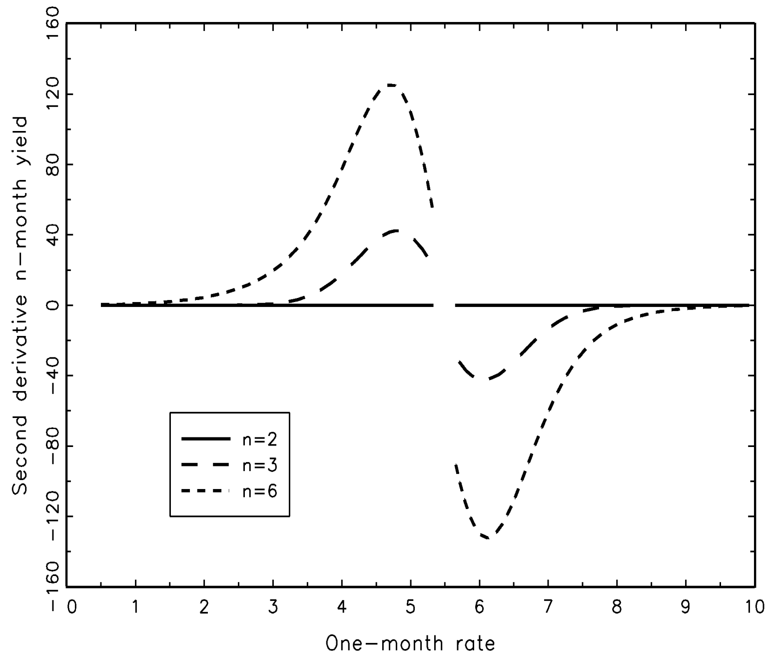

Figure 3: Second derivative of the yield function against the short rate.

For $n = 2$ the function is identically zero since $f_2(x)$ is stepwise linear, so the second derivative disappears at all points of continuity. For $n = 3$ and $n = 6$ the figure shows that there is in fact a nonlinearity around the threshold value (that could not be made visible in figure 2 for these small times to maturity). In particular, the yield functions f_3 and f_6 exhibit a convex-concave pattern: on the left of the threshold value the sensitivity of y^n with respect to x increases (positive second derivative, i.e. increase in (positive) first derivative), on the right it decreases (negative second derivative, i.e. decrease in (positive) first derivative). Moreover, the interval in which 'nonlinearity prevails' is wider for $n = 6$ than for $n = 3$.

Finally, we can use our analytical formula for arbitrage-free bond yields to assess the effect of changes in the key parameters of short-rate dynamics on the

[15] That is, we do not compute $k_n(x)$ if $[x-h, x+h]$ contains the threshold value c.

shape of the yield function. For instance, an increase of β in (2.2) induces an increase in the size of the 'jump' at the threshold value. Hence, with reference to figure 2, the parameter change shifts up the yield function to the right of the threshold value but hardly alters it to the left of it. Moreover, the nonlinearity becomes more distinct, which would be reflected by an amplification of the sine-shaped pattern in figure 3. The only impact of increasing the threshold value c is a shift of the yield function 'within itself': the jump of the yield function shifts to the right, but apart from that, the position of the function is unaffected. The main effect of an increase in the persistence parameter κ is an increase in the slope of the yield function. Concerning second derivatives, the convex-concave pattern becomes more distinct.

5 Summary and Outlook

While the large empirical literature devoted to modeling and estimating short-term interest rate dynamics is bringing up increasingly rich and advanced specifications, the literature on arbitrage-free term structure models usually restricts itself to simple linear state dynamics. This is because "researchers are inevitably confronted with trade-offs between the richness of econometric representations of the state variables and the computational burdens of pricing and estimation" as Dai and Singleton (2000), p. 1943, observe.

We have derived the complete arbitrage-free term structure for one particular case of nonlinear state dynamics: the short-term interest rate follows a self-exciting threshold autoregressive (SETAR) process expounded by Lanne and Saikkonen (2002), which proves empirically successful. In the version considered here, the one-month rate follows an autoregressive process, for which the intercept is allowed to switch between two regimes. When the previous period's short rate has exceeded a certain threshold value, the high intercept prevails, otherwise, the intercept assumes its lower value.

The derived yield function, mapping the one-month rate into bond yields of longer maturity, exhibits a convex-concave pattern around the threshold value. For short time to maturity n, the corresponding yield as a function of the short rate is nearly (stepwise) linear. For longer maturities, the relationship between the short rate and the long-term rate is convex for low levels of the short rate and concave for high levels. This pattern can also be found in US yield data.

Our analysis confirms that the trade-off asserted by Dai and Singleton (2000) does apply to the threshold model considered here as well, unfortunately. On the one hand, compared to purely linear models, the threshold dynamics allow more flexibility in modeling the time-series properties of the short rate. On the other hand, however, computing long-term bond yields becomes a more intricate task. Although our solution for bond prices delivers the exact solution in a finite number of operations, two numerical problems arise for the computation of yields with a longer time to maturity (i.e. more than nine months). First, the solution for the n-period yield requires the computation of the c.d.f. of an $(n-2)$-variate normal with general variance-covariance matrix, which is usually not a simple task for standard statistical software packages if the dimension of the multivariate normal is large. The second problem lies in the fact that the computational burden increases exponentially with time to maturity. A promising solution for the first problem is the application of numerical methods (e.g. the GHK-simulator) to compute the required cumulative distribution functions. Concerning the curse-of-dimensionality problem, one may employ a mixed approach that makes use of our analytical solution within a simulation-based approach.

As an application, it would be interesting to employ the model for pricing interest rate derivatives. Exploring the implications for risk management seems to be another fruitful avenue of further research: in contrast to models from the affine class, the threshold model implies that the sensitivity of long-term yields to changes in the short rate is not constant but changes with its level. This has a direct impact on risk measures.

The solution approach introduced in this paper is likely to be transferable to richer nonlinear models of the term structure. For instance, parameters other than intercepts may be allowed to switch as well. Moreover, the technique introduced in this paper is also applicable for multifactor models.

Appendix

A Two Auxiliary Results

We first provide two auxiliary results about the expectations of the (truncated) multivariate log-normal distribution. Let x be distributed as an m-variate normal, $x \sim N(0, \Omega)$, and let d and r be vectors of length m. Then:

$$E(\exp[d'x]) = \exp\left[\frac{1}{2}d'\Omega d\right] \tag{A.1}$$

and

$$E(\exp[d'x]|x < r) = \frac{1}{Pr(x < r)} F(r; \Omega d, \Omega) \exp\left[\frac{1}{2}d'\Omega d\right], \tag{A.2}$$

where $F(r; \Omega\, d, \Omega)$ denotes the c.d.f. of the multivariate normal with mean Ωd and variance-covariance matrix Ω evaluated at r.

Noting that $d'x$ is a scalar normal random variable, the first expression is a standard result. To show the second result, first note that the conditional density required to compute the expectation is given by

$$p(x|x < r) = \frac{p(x)}{Pr(x < r)}.$$

where $p(x)$ is the density of the normal $N(0, \Omega)$ and $Pr(x < r)$ is the c.d.f. of that normal evaluated at r.[16]

Then we have

$$\begin{aligned}
& E(\exp[d'x]|x < r) \\
= & \frac{1}{Pr(x < r)} \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \exp[-1/2x'\Omega^{-1}x]\ \exp[d'x]\, dx_1 \ldots dx_m \\
= & \frac{1}{Pr(x < r)} \cdot \exp[(1/2)d'\Omega d] \cdot \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \\
& \times \exp[(-1/2)(x - \Omega d)'\Omega^{-1}(x - \Omega d)]\, dx_1 \ldots dx_m \\
= & \frac{1}{Pr(x < r)} \cdot \exp[(1/2)d'\Omega d] \cdot F(r; \Omega d, \Omega).
\end{aligned}$$

B Derivation of the Bond Pricing Formula for $n > 2$

1. Representation of X_{t+i} and partial sums of log SDFs

In the following we will need X_{t+i} written in terms of X_t, future ϵ_t and future intercepts as well as partial sums of the pricing kernel M_t.

[16] So we could write here and in (A.2) $F(r; 0, \Omega)$ instead of $Pr(x < r)$. However, we stick to $Pr(x < r)$ since this is a more convenient notation for the derivation following in section B.

Starting with X_t and iterating (3.9) forward leads to

$$\begin{aligned} X_{t+i} &= \kappa^i X_t + \kappa^{i-1} a(S_t) + \kappa^{i-2} a(S_{t+1}) + \ldots + \kappa a(S_{t+i-2}) + a(S_{t+i-1}) \\ &\quad + \sigma\kappa^{i-1}\epsilon_{t+1} + \sigma\kappa^{i-2}\epsilon_{t+2} + \ldots + \sigma\kappa\epsilon_{t+i-1} + \sigma\epsilon_{t+i}, \end{aligned}$$

in compact form

$$X_{t+i} = \kappa^i X_t + \sum_{l=1}^{i} g_l^i a(S_{t+l-1}) + h_l^i \epsilon_{t+l}. \tag{B.3}$$

The sum of the log discount factors, $m_t = \ln(M_t)$, can be written as

$$\begin{aligned} & m_{t+1} + m_{t+2} + \ldots + m_{t+n} \\ = & -n\delta - X_t - X_{t+1} - \ldots - X_{t+n-1} - \sigma\lambda\epsilon_{t+1} - \sigma\lambda\epsilon_{t+2} - \ldots - \sigma\lambda\epsilon_{t+n} \\ = & -n\delta - (1 + \kappa + \ldots + \kappa^{n-1}) X_t \\ & -(1 + \kappa + \ldots + \kappa^{n-2}) a(S_t) - (1 + \kappa + \ldots + \kappa^{n-3}) a(S_{t+1}) - \ldots \\ & -(1 + \kappa) a(S_{t+n-3}) - a(S_{t+n-2}) \\ & -\sigma(\lambda + 1 + \kappa + \ldots + \kappa^{n-2})\epsilon_{t+1} - \sigma(\lambda + 1 + \kappa + \ldots + \kappa^{n-2})\epsilon_{t+2} - \ldots \\ & -\sigma(\lambda + 1)\epsilon_{t+n-1} - \sigma\lambda\epsilon_{t+n-2}, \end{aligned}$$

compactly,

$$m_{t+1} + \ldots + m_{t+n} = -n\delta - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i} + \sum_{j=0}^{n-2} c_j^n \cdot a(S_{t+j}). \tag{B.4}$$

2. Bond price as product of three factors

We plug (B.4) into the bond price formula (3.2) and obtain

$$\begin{aligned} P_t^n &= E(\exp[m_{t+1} + \ldots + m_{t+n}] | X_t) \\ &= E\left(\exp\left[-n\delta - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i} + \sum_{j=0}^{n-2} c_j^n \cdot a(S_{t+j}) \right] \middle| X_t \right). \end{aligned}$$

The random variables X_t and S_t are part of the conditioning information set and can thus be taken outside the expectation. (Note that knowing X_t implies knowing if $X_t < c$ is true and thus knowing the realization of $S_t = I(X_t \geq c)$.) Moreover, ϵ_{t+n-1} and ϵ_{t+n} are independent of $(S_t, S_{t+1}, S_{t+n-2}, \epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$. Hence, we can write

$$\begin{aligned} P_t^n &= \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \\ &\times E(\exp[b_{n-1}^n \epsilon_{t+n-1} + b_n^n \epsilon_{t+n}] | X_t) \\ &\times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i}) \right] \middle| X_t \right). \end{aligned} \tag{B.5}$$

The product consists of three factors. The first factor contains only quantities known at time t. The expectation of the second factor can be computed using the first of our auxiliary results, (A.1), since $(\epsilon_{t+n-1}, \epsilon_{t+n})'$ is conditionally (and unconditionally) normally distributed. Thus, using the terms of (A.1) we have $d = (b^n_{n-1}, b^n_n)'$, $x = (\epsilon_{t+n-1}, \epsilon_{t+n})'$, $\mu = 0_2$, and $\Omega = I_2$ and we obtain for the second factor in (B.5)

$$E(\exp[b^n_{n-1}\epsilon_{t+n-1} + b^n_n\epsilon_{t+n}]|X_t) = \exp[0.5(b^n_{n-1})^2 + 0.5(b^n_n)^2]. \qquad \text{(B.6)}$$

3. Computation of $E\left(\exp\left[\sum_{i=1}^{n-2} b^n_i\epsilon_{t+i} + \sum_{j=1}^{n-2} c^n_j a(S_{t+j})\right]\Big| X_t\right)$.

For computing the third factor in (B.5) it is important to note that $(S_{t+1}, \ldots, S_{t+n-2})'$ and $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ are not independent. We will evaluate the expression by first computing the expectation for an arbitrary *given* realization of $(S_{t+1}, \ldots, S_{t+n-2})'$ and then take the probability-weighted sum over all possible realizations of $(S_{t+1}, \ldots, S_{t+n-2})'$. That is, we first enlarge the conditioning information set and then integrate out the enlargement again.[17]

Let

$$\bar{\zeta}_t = (\bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2})'$$

denote a realization of

$$\zeta_t = (S_{t+1}, \ldots, S_{t+n-2})',$$

i.e $\bar{\zeta}_t$ is a sequence of zeros and ones. There are 2^{n-2} different such sequences. They will be indexed $k = 1, 2, \ldots, 2^{n-2}$ such that $k-1$ is that decimal number that corresponds to the binary number represented by $\bar{\zeta}_t$. For example for $n = 6$, $\bar{\zeta}_t(k = 1) = (0, 0, 0, 0)'$ and $\bar{\zeta}_t(k = 14) = (1, 1, 0, 1)'$.

Thus, we have

$$\begin{aligned} & E\left(\exp\left[\sum_{i=1}^{n-2} b^n_i\epsilon_{t+i} + c^n_i a(S_{t+i})\right]\Bigg| X_t\right) \\ = & \sum_{k=1}^{2^{n-2}} Pr\big(\bar{\zeta}_t(k)\big|\, X_t\big)\ E\left(\exp\left[\sum_{i=1}^{n-2} b^n_i\epsilon_{t+i} + c^n_i a(\bar{S}_{t+i}(k))\right]\Bigg| X_t, \bar{\zeta}_t(k)\right) \end{aligned}$$

where $Pr(\bar{\zeta}_t(k)|X_t)$ denotes the conditional probability of the realization $\zeta_t = \bar{\zeta}_t(k)$.

For the expectation conditional on the augmented information set we can pull

[17] A similar approach is taken by Bansal and Zhou (2002), deriving bond prices for the case that the state evolution is subject to Markov regime switching.

out expressions involving $\bar{S}_{t+i}$, hence

$$
\begin{aligned}
& E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\Big| X_t\right) \\
= & \sum_{k=1}^{2^{n-2}} Pr\big(\bar{\zeta}_t(k)\big| X_t\big) \exp\left[\sum_{i=1}^{n-2} c_i^n a(\bar{S}_{t+i}(k))\right] \\
& \times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i}\right]\Big| X_t, \bar{\zeta}_t(k)\right). \qquad \text{(B.7)}
\end{aligned}
$$

4. Computation of $E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i}\right]\Big| X_t, \bar{\zeta}_t(k)\right)$.
In order to compute the last conditional expectation appearing in the latter expression we will make use of our auxiliary result (A.2). For this we will rewrite the conditioning information set as a set of inequality conditions.

To explain the approach, we first consider the following example. If, for $n = 5$, $\bar{\zeta}_t(3) = (0, 1, 0)$, this is equivalent to the event

$$X_{t+1} < c, X_{t+2} \geq c, X_{t+3} < c.$$

Making use of (B.3), these three inequalities can be written as

$$
\begin{aligned}
\kappa X_t + g_1^1 a(S_t) + h_1^1 \epsilon_{t+1} &< c \\
\kappa^2 X_t + g_1^2 a(S_t) + a_2^2 a(\bar{S}_{t+1}(k)) + h_1^2 \epsilon_{t+1} + h_2^2 \epsilon_{t+2} &\geq c \\
\kappa^3 X_t + g_1^3 a(S_t) + g_2^3 a(\bar{S}_{t+1}(k)) + g_3^3 a(\bar{S}_{t+2}(k)) + h_1^3 \epsilon_{t+1} + h_2^3 \epsilon_{t+2} + h_3^3 \epsilon_{t+3} &< c
\end{aligned}
$$

To be able to apply our auxiliary result (A.2) we only want to have '<' inequalities. So we multiply every '≥' inequality by -1. Technically, we multiply through any inequality by a factor $R(\bar{S}_{t+i}(k))$, where for the function $R(\cdot)$ defined on $\{0, 1\}$, $R(0) = 1$, and $R(1) = -1$. Hence, in the above example $R(\bar{S}_{t+1}(k)) = R(0) = 1$, $R(\bar{S}_{t+2}(k)) = -1$, and $R(\bar{S}_{t+3}(k)) = 1$. Thus, the inequality corresponding to a particular $\bar{S}_{t+i}(k)$ is written as

$$R(\bar{S}_{t+i}(k)) \cdot \left[\kappa^i X_t + \sum_{l=1}^{i} g_l^i a(\bar{S}_{t+l-1}) + h_l^i \epsilon_{t+l}\right] < R(\bar{S}_{t+i}(k))c. \qquad \text{(B.8)}$$

Accordingly, the set of inequalities corresponding to a particular $(\bar{S}_{t+1}(k), \ldots, \bar{S}_{t+n-2}(k))'$ can be written in vector-matrix notation as

$$\begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ R(\bar{S}_{t+2}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix} \odot \left(\begin{pmatrix} \kappa \\ \kappa^2 \\ \vdots \\ \kappa^{n-2} \end{pmatrix} X_t \right.$$

$$+ \begin{pmatrix} g_1^1 & 0 & 0 & \dots & 0 \\ g_1^2 & g_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1^{n-2} & g_2^{n-2} & g_3^{n-2} & \dots & g_{n-2}^{n-2} \end{pmatrix} \begin{pmatrix} a(S_t) \\ a(\bar{S}_{t+1}(k)) \\ \vdots \\ a(\bar{S}_{t+n-3}(k)) \end{pmatrix}$$

$$\left. + \begin{pmatrix} h_1^1 & 0 & 0 & \dots & 0 \\ h_1^2 & h_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{n-2} & h_2^{n-2} & h_3^{n-2} & \dots & h_{n-2}^{n-2} \end{pmatrix} \begin{pmatrix} \epsilon_{t+1} \\ \epsilon_{t+2} \\ \vdots \\ \epsilon_{t+n-2} \end{pmatrix} \right)$$

$$< \ c \cdot \begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ R(\bar{S}_{t+2}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix}$$

where '$\odot$' denotes elementwise multiplication of two vectors. Using the definitions (3.17) - (3.26), and $\mathcal{E}_t = (\epsilon_{t+1}, \dots, \epsilon_{t+n-2})'$ this is written compactly as

$$\tilde{f}(k)X_t + \tilde{G}(k)a(\bar{\zeta}_t^*(k)) + \tilde{H}(k)\mathcal{E}_t < \tilde{c}(k) \tag{B.9}$$

or

$$\tilde{H}(k)\mathcal{E}_t < \tilde{h}(k). \tag{B.10}$$

It is important to note that multiplying both sides of (B.10) by the inverse of $\tilde{H}(k)$ would not be an equivalent transformation of that inequality.[18] We define a new random vector

$$\tilde{z}(k) = \tilde{H}(k)\,\mathcal{E}_t.$$

Since $\mathcal{E}_t \sim N(0_{n-2}, I_{n-2})$, we have

$$\tilde{z}(k) \sim N(0, \tilde{H}(k)\tilde{H}(k)')$$

Now we can turn the expression to be computed,

$$E\left(\exp\left[b^{*\prime}\mathcal{E}_t\right] \middle| X_t, \bar{\zeta}_t(k)\right),$$

[18] As a simple example, one can easily verify that $Ax < c$ – with $A = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}$, $x = (x_1, x_2)'$, $c = (c_1, c_2)'$, a_1, a_2, a_3, c_1, c_2 all positive – defines a different region in (x_1, x_2)-space than $x < A^{-1}c$.

into the form of (A.2).[19] That is we rewrite the exponential in terms of $\tilde{z}(k)$ and the conditioning on $\bar{\zeta}_t(k)$ in terms of an inequality for $\tilde{z}(k)$. Then we apply (A.2). We obtain

$$\begin{aligned}
& E\left(\exp\left[b^{*\prime}\mathcal{E}_t\right]\middle| X_t, \bar{\zeta}_t(k)\right) \\
= \ & E\left(\exp\left[(\tilde{H}(k)^{-1\prime}b^*)'\tilde{z}(k)\right]\middle| X_t,\, \tilde{z}(k) < \tilde{h}(k)\right) \\
= \ & \frac{1}{Pr(\tilde{z}(k) < \tilde{h}(k)|X_t)} \times \exp\left[0.5\, b^{*\prime}b^*\right] \times F\left(\tilde{h}(k); \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}$$

Finally note that

$$Pr(\tilde{z}(k) < \tilde{h}(k)|X_t) = Pr(\bar{\zeta}_t(k)|X_t),$$

since $\{\tilde{z}(k) < \tilde{h}(k)|X_t\}$ and $\{\bar{\zeta}_t(k)|X_t\}$ are equivalent events as we derived above.

5. Putting things together

In step 4 we computed the last term in (B.7). Plugging in we obtain

$$\begin{aligned}
& E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\middle| X_t\right) \\
= & \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] \\
& \times \exp\left[0.5\, b^{*\prime}b^*\right] \cdot F\left(\tilde{h}(k); \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}$$

Using the latter and (B.6) we obtain for the bond price (B.5),

$$\begin{aligned}
P_t^n \ = \ & \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \\
& \times E(\exp[b_{n-1}^n \epsilon_{t+n-1} + b_n^n \epsilon_{t+n}]|X_t) \\
& \times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\middle| X_t\right) \\
= \ & \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \cdot \exp[0.5(b_{n-1}^n)^2 + 0.5(b_n^n)^2] \\
& \times \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] \\
& \times \exp\left[0.5\, b^{*\prime}b^*\right] \cdot F\left(\tilde{h}(k); \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\right) \\
= \ & \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \cdot \exp[0.5\, b'b] \\
& \times \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] F\left(\tilde{h}(k); \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}$$

Transferring the price into a yield using (2.3) completes the proof.

[19] Note that the only slight difference to (A.2) is that everything is conditional on X_t. However, a 'conditional version' of (A.2) could be derived in the same way as the unconditional version.

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