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BLACK-SCHOLES THEORY FOR AN UNDERLYING WITH MULTIPLE ATTRACTORS

FREDERIK HERZBERG

ABSTRACT. A valuation theory for derivatives on an underlying that is subject to multiple attractors is proposed, the economic justification being attraction-adjusted hedging. In non-critical regions – outside the boundaries of the attractor regions – a European option price can be viewed as a derivative on an underlying with a mean-reverting law, such as a commodity price, however with a different payoff function.

1. INTRODUCTION

In recent years, numerous authors in the macroeconomic theory of financial markets have been studying multiple attractor regimes for both traded and non-traded assets, with a particular emphasis on currencies, aiming at rigorous economic explanations for currency crises. Starting as early as 1986, Obstfeld introduced the first model to explain currency crises [17], followed by the seminal work of Morris and Shin [16] which then prompted many other researchers to propose so-called "second generation models", most notably Jeanne and Masson [12] (see also the survey article by Jeanne [11]), and also sparked some more empirical research such as the work by Sarno and Taylor [20]. More recently still, after fixed (pegged) exchange rate regimes seemed to be the focus of research in that area, Jeanne and Rose [10] shifted the attention back to traded assets and floating exchange rate regimes. At the heart of the aforementioned "second generation" investigations lies the relation between multiple – "sunspot", as opposed to economically justified, – equilibria and self-replicating – or *contagious*, as it is often referred to – behaviour.

On a different front of research and in a very different methodological manner, motivated by stochastic physical systems, stochastic models for herd behaviour and self-fulfilling prophecies have been developed, e.g. by Corcos *et al.* [6].

Furthermore, empirical research on the joint evolution of stock and futures prices, as well as on the joint evolution of spot and forward exchange rates, has been conducted by Sarno and Valente [19], as well as by Clarida, Sarno, Taylor and Valente [5], respectively. These papers propose regime-switching vector equilibrium models.

The problem of how to economically explain price formation for derivatives on traded assets with multiple equilibrium prices, however, has received little attention so far.

In this paper, we will therefore develop a Black-Scholes type theory for European derivatives with an underlying whose price process has multiple attractors which attract at linear speed: Based on a hedging argument, we will derive a partial

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differential equation for a European option price and solve it probabilistically, the solution being the conditional expectation of the discounted payoff function.

Finally, we shall explain why under the assumptions of our model, the derivative price coincides locally with the price for a derivative on an underlying whose price process has a mean-reverting drift, for instance when much of the market price is actually determined by the fundamental economic factors of supply and demand. This, of course, is a model assumption that was formulated for the first time in the theory of commodity derivatives by Fischer Black [3], and later generalised by Miltersen and Schwartz [15] to even incorporate a given term structure of interest rates in the spririt of Heath, Jarrow and Morton [7, 8] as well as convenience yields.

2. Description of the model

Consider the simplest non-trivial market model, in which there is only one asset A, and the bond B, yielding interest at rate $r > 0$.

Let $\varepsilon > 0$, and let the stochastic process $x^{(\varepsilon)} = \left(x_t^{(\varepsilon)} \right)_t$ denote the logarithmic price process of an asset A that is subject to a multiple attractor regime, the attraction occurring at linear (in the log-price) speed whenever an ε-ball (wherein $\varepsilon > 0$ shall be conceived of as the reaction threshold) around the current logarithmic price is inside some attractor region, and the difference between the log-price process and this attraction term is assumed to follow a Black-Scholes model with risk-less rate r, i.e. is just Brownian motion with constant volatility $\sigma > 0$ and drift $r - \frac{\sigma^2}{2}$. For the following, we shall drop the superscript ε where no ambiguity can arise.

In addition, we also assume that, given some derivative D on the underlying asset A, any previsible and self-financing portfolio process (in the sense of, e.g., Karatzas [13]) in A, D and the bond, must grow at the weighted mean of the risk-less rate r (for the bond and derivative parts) and the attraction-induced rate $r - \sum_{j=1}^N \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j (m_j - x_t)$. (In case $\nu_1, \ldots, \nu_n = 0$, we simply get the usual Black-Scholes model.)

Put formally, this is to say that in our simplified market model we assume the following (*existence of multiple linearly attracting equilibria*):

There are

- a natural number $N > 1$ (the number of *attractors*),
- N different *attractors* (or *equilibrium levels*) $m_1, \ldots, m_N \in \mathbb{R}$, enumerated in increasing order,
- *attractor regions* $A_1, \ldots, A_N$ (pairwise disjoint, finite or infinite, left-open intervals $\subset \mathbb{R}$, by assumption on $\vec{m}$ also increasing), and
- *attraction intensities* $\nu_1, \ldots, \nu_N > 0$,

such that

$$(1) \qquad\qquad\qquad\qquad \forall j \in \{1, \ldots, N\} \quad m_j \in A_j,$$

$$(2) \qquad\qquad\qquad\qquad \bigcup_{i=1}^N A_i = \mathbb{R},$$

$$(3) \quad \forall t > 0 \quad dx_t = \left(r - \frac{\sigma^2}{2} \right) dt + \sum_{j=1}^N \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j (m_j - x_t) \ dt + \sigma \ db_t,$$

(wherein b is the – normalised – Wiener process) and furthermore – assuming $\varepsilon > 0$ to be sufficiently small – the functions $\chi_{A_j}^{(\varepsilon)}$, $j \in \{1, \ldots, M\}$, are Lipsschitz and

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coincide with χ_{A_j} outside an ε-nehighbourhood of the boundary of A_j, e.g.

$$\chi_{A_j}^{(\varepsilon)} : x \mapsto \left(\frac{x - a_{i-1}}{\varepsilon}\right) \chi_{(a_{i-1}, a_{i-1}+\varepsilon]}(x) + \chi_{(a_{i-1}+\varepsilon, a_i-\varepsilon]}(x)$$
$$+ \left(\frac{a_i - x}{\varepsilon}\right) \chi_{(a_i-\varepsilon, a_i]}(x)$$

for $1 < i < M$ and $A_i = (a_{i-1}, a_i]$, as well as

$$\chi_{A_M}^{(\varepsilon)} : x \mapsto \left(\frac{x - a_{M-1}}{\varepsilon}\right) \chi_{(a_{M-1}, a_{M-1}+\varepsilon]}(x) + \chi_{(a_{M-1}+\varepsilon, +\infty)}(x)$$

and

$$\chi_{A_1}^{(\varepsilon)} : x \mapsto \chi_{(-\infty, a_1-\varepsilon]}(x) + \left(\frac{a_1 - x}{\varepsilon}\right) \chi_{(a_1-\varepsilon, a_1]}(x).$$

Any trader who is aware of the multiplicity of equilibria will have to adjust her portfolio accordingly. Since there is no previsible portfolio that would eliminate all stochasticity (such as a perfect Δ-hedge), we cannot hope to find an objective, as it were, risk-neutral hedge. Thus, any self-financing previsible portfolio π_t will grow at a stochastic rate $d\pi_t$ for any $t \geq 0$.

Nevertheless, one can ask how a representative agent who knows about the multiplicity of the equilibria would valuate a previsible self-financing portfolio which contains α_t shares of the asset A at time t. A representative agent will expect the portfolio to grow essentially at the risk-neutral rate, except that the attraction term in (3) must be taken into account (and weighted with the number of shares α_t of asset A):

In order to see this, note that by Itô's formula,

$$\forall t > 0 \quad d\left(e^{x_t}\right) = re^{x_t} \, dt + e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right) \, dt + e^{x_t} \sigma \, db_t,$$

whence we obtain as the mean growth rate of e^{x_t} the value

$$\left.\frac{d}{du}\right|_{u=t} \mathbb{E}\left[e^{x_u} \mid \mathcal{F}_t\right] = \left(r + \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right)\right) e^{x_t},$$

whereas if x had been the logarithmic price process of the Black-Scholes model, this value would, of course, have been $\left.\frac{d}{du}\right|_{u=t} \mathbb{E}\left[e^{x_u} \mid \mathcal{F}_t\right] = re^{x_t}$.

Hence, in expectation, the assets in the portfolio do not grow at the risk-neutral rate $re^{x_t} \, dt$ for a given time $t > 0$, but at the rate of $\left(r + \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right)\right) e^{x_t} \, dt$. Therefore, the portfolio growth in our model equals the portfolio growth in the Black-Scholes model (viz. at the risk-neutral rate $r\pi_t \, dt$) plus $e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right) \, dt$ for each unit of asset A in the portfolio.

Since by assumption there are α_t shares of asset A in the portfolio, we get the following stochastic differential equation for the portfolio value:

$$\forall t > 0 \quad d\pi_t = r \cdot \left(\alpha_t e^{x_t} + \beta_t + \delta_t v(x_t, t)\right) \, dt$$
$$+ \alpha_t e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right) \, dt$$
$$= r\pi_t \, dt + \alpha_t e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j \left(m_j - x_t\right) \, dt.$$

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This motivates the following Definition:

Definition 2.1. *Consider a portfolio $(\alpha_t, \beta_t, \delta_t)_t$ consisting of α_t units of asset A, β_t units of the bond B and δ_t contracts of the derivative D with maturity $T > 0$ (whose value at time t is assumed to only depend on x_t and shall be denoted by $v(x_t, t)$), and assume that the portfolio $(\alpha_t, \beta_t, \delta_t)_t$ is previsible and self-financing. The portfolio will be called an* attraction-adjusted hedging portfolio for the derivative D *(for short,* attraction-adjusted*) if and only if its value π_t at time t will satisfy the stochastic differential equation*

$$(4) \qquad \forall t > 0 \quad d\pi_t = r \cdot \left(\alpha_t e^{x_t} + \beta_t + \delta_t v(x_t, t) \right) \, dt$$
$$+ \alpha_t e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}(x_t) \cdot \nu_j (m_j - x_t) \, dt.$$

3. Consistency of the model

In order to vindicate the first assumption (most notably (3)) mathematically, let us remark that any initial value problem based on the stochastic differential equation (3) (that is the problem of solving (3) subject to the condition that $x_0 = \xi_0$ $\mathbb{P}$-a.s. for some real number ξ_0) has, according to well-known results (cf e.g. Arnold [1] or Revuz and Yor [18]) a unique solution, since $\chi_{A_1}^{(\varepsilon)}, \ldots, \chi_{A_M}^{(\varepsilon)}$ and thereby all the coefficients of (3) are Lipschitz.

Remark 3.1. *By Girsanov's Theorem, there is for all $x \in \mathbb{R}$ (start log-price) a probability measure $\mathbb{Q}^x$, equivalent to $\mathbb{P}^x$, under which $(x_t)_t$ – after adding a linear drift at rate $r - \frac{\sigma^2}{2}$ – is just Brownian motion with volatility σ, hence $\left(\exp\left(x_t - \left(r - \frac{\sigma^2}{2} \right) t \right) \right)_t$ becomes a martingale under $\mathbb{Q}^x$ for all $x \in \mathbb{R}$. Therefore, again referring to Karatzas [13], we may conclude that there is no arbitrage in this market model.*

4. Attraction-adjusted European derivative prices

Consider an investor that has issued a derivative contract of type D (whose value at any time t and current underlying log-price $x.$, as stipulated previously, shall be denoted $v(x, t)$ for any time t and current underlying log-price x).

In order to Δ-hedge his position, the investor will set up a portfolio whose value, conceived of as a stochastic process $(\pi_t)_t$, satisfies the stochastic differential equation

$$(5) \qquad \forall t > 0 \quad d\pi_t = - \, dv(e^{x_t}, t) + (\partial_1 v(e^{x_t}, t)) \, d(e^{x_t}).$$

Now, to some extent analogously to the derivation and solution of the classical Black-Scholes partial differential equation [4], we may prove the following formula for a European derivative:

Theorem 4.1. *Under the assumptions of Section 2, there is a unique function $v : \mathbb{R}_{>0} \times [0, T] \to \mathbb{R}$, such that the portfolio of value $\pi.$, i.e. the Δ-hedge for D, becomes attraction-adjusted in the sense of Definition 2.1. If we choose g such that $v(\cdot, T) = g(\ln(\cdot))$, then v is given by the formula*

$$(6) \qquad \forall x \in \mathbb{R} \, \forall t \in [0, T] \quad v(e^x, t) = \mathbb{E}^x \left[e^{-r(T-t)} g(x_{T-t}) \right],$$

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and the function $u : (x, t) \mapsto v(e^{x_t}, t)$ solves the terminal value problem

$$(7) \qquad \frac{\sigma^2}{2} \Delta u + \partial_2 u + r \partial_1 u + \left(\sum_{j=1}^N \chi_{A_j}^{(\varepsilon)} ((\cdot)_1) \cdot \nu_j (m_j - (\cdot)_1) \right) \partial_1 u - r u = 0$$

$$\text{on } \mathbb{R} \times (0, T),$$
$$u(\cdot, T) = g \qquad \text{on } \mathbb{R}.$$

Furthermore, any derivative without dividends will solve the above partial differential equation.

The proof will be given in an Appendix.

The characterisation of the fair derivative price – "fair" in the sense of covering the cost for the hedge – as an expected discounted payoff has – apart from its theoretical appeal – some practically relevant consequences, for instance put-call parities. Furthermore, the partial differential equation in the Theorem can be used to show that any attraction-adjusted European derivative price coincides locally with the price for some commodity derivative price with a suitable payoff function:

Remark 4.1. *Inside $(a_i + \varepsilon, a_{i+1} - \varepsilon) \times (0, T)$, for all $i < M$, the partial differential equation obeyed by u is exactly the same that $\bar{u}^{(i)} : (x, t) \mapsto \mathbb{E}^x \left[e^{-r(T-t)} g \left(y_{T-t}^{(i)} \right) \right]$, wherein $y_\cdot^{(i)}$ is an Ornstein-Uhlenbeck (that is, mean-reverting) process of reverting speed ν_i and mean m_i, would follow:*

$$(8) \qquad \frac{\sigma^2}{2} \Delta u^{(i)} + \partial_2 u^{(i)} + r \partial_1 u^{(i)} + \left(\partial_1 u^{(i)} \right) \chi_{A_i}^{(\varepsilon)} ((\cdot)_1) \cdot \nu_i (m_i - (\cdot)_1) - r u^{(i)} = 0$$

$$\text{on } (a_i + \varepsilon, a_{i+1} - \varepsilon) \times (0, T),$$

which (due to $\chi_{A_i}^{(\varepsilon)} = 1$ on $(a_i + \varepsilon, a_{i+1} - \varepsilon)$) is the same as

$$(9) \qquad \frac{\sigma^2}{2} \Delta u^{(i)} + \partial_2 u^{(i)} + r \partial_1 u^{(i)} + \left(\partial_1 u^{(i)} \right) \cdot \nu_i (m_i - (\cdot)_1) - r u^{(i)} = 0$$

$$\text{on } (a_i + \varepsilon, a_{i+1} - \varepsilon) \times (0, T).$$

Therefore, on each of the regions $(a_i + \varepsilon, a_{i+1} - \varepsilon) \times (0, T)$, u will solve the same partial differential equation as the price for a derivative whose underlying is modelled by a logarithmic Ornstein-Uhlenbeck process[1] with mean-reversion speed ν_i and mean m_i.

A commodity derivative (with suitable payoff function to fit the Dirichlet boundary data of $u^{(i)}$) may serve as an example for such a derivative whose logarithmic underlying price follows an Ornstein-Uhlenbeck process: Commodity prices are often modelled as having a mean-reverting law (cf Black [3] and more recently Miltersen and Schwartz [15]).

It might be possible to obtain results on the values of American derivatives in this setting as well:

Remark 4.2. *Since Theorem 4.1 contains a Black-Scholes type partial differential equation (7) for arbitrary derivatives — however, for simplicity, we did not allow*

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[1] Recall that an Ornstein-Uhlenbeck process with mean-reversion speed ν_i and mean m_i is the solution $y_\cdot$ to the stochastic differential equation

$$(10) \qquad dy_t = \left(r - \frac{\sigma^2}{2} \right) dt + \nu_i (m_i - y_t) \, dt + \sigma \, db_t$$

for all $t > 0$.

for dividend yields — , any American-European difference will also solve the partial differential equation (7). Hence, in order to approximate the difference between American and European prices. Now, mimicking the technique of MacMillan [14] and of Barone-Adesi and Whaley [2] (as summarised e.g. by Hull [9]), one can introduce appropriate changes of variables and drop a "small" term, to obtain an analytically more tractable partial differential equation. This may, possibly after further approximations, lead to an analytic approximation of the American-European difference in a multiple-attractor regime.

5. Conclusion

Even under the assumption of *multiple* equilibrium prices – conceived of as multiple attractors – for an underlying asset, much of the classical Black-Scholes theory can be saved. However, rather than being risk-eliminating as in the Black-Scholes world, the appropriate hedging in a multiple equilibrium setting can only be *attraction-adjusted*.

For the special case of a portfolio consisting only of underlying stock and one derivative contract, the attraction-adjusted hedging strategy gives rise to a partial differential equation for the "fair price" (in the sense of covering the cost of the hedge) of the derivative. Solving the corresponding terminal value problem for a European option probabilistically, a valuation formula can be deduced. The option price then is simply the expected discounted payoff under the *attraction-adjusted* probability measure – the risk-neutral measure cannot be taken, due to the multiplicity of attractors that entail pro-cyclic behaviour.

Finally, the European option price thus obtained can be viewed as the price for some commodity derivative (commodities being assumed to have a mean-reverting law) with a suitable payoff function, as locally the former and the latter derivative prices obey the same partial differential equation.

Appendix A. Proof of the Theorem

Proof of Theorem 4.1. First, we use Itô's formula to prove

$$(11) \qquad \forall t > 0 \quad d\left(e^{x_t}\right) = re^{x_t} \, dt + e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left(x_t\right) \cdot \nu_j \left(m_j - x_t\right) \, dt + e^{x_t} \sigma \, db_t$$

and also

$$(12) \qquad \forall t > 0 \quad dv\left(e^{x_t}, t\right)$$

$$= \left(\partial_1 v\left(e^{x_t}, t\right)\right) \cdot e^{x_t} \left(r \, dt + \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left(x_t\right) \cdot \nu_j \left(m_j - x_t\right) \, dt + \sigma \, db_t \right)$$

$$+ \partial_2 v\left(e^{x_t}, t\right) \, dt + \frac{\sigma^2}{2} e^{2x_t} \Delta v\left(e^{x_t}, t\right)$$

(where, of course, $\Delta = \partial_1 \partial_1$)

This implies that the portfolio of value π. set up by the derivative-issuer to Δ-hedge his position is indeed previsible and therefore, since it is also self-financing (any gain or loss made on the derivative will be compensated by a loss or gain in the asset, respectively), our second assumption in the guise of (4) may be employed.

Hence, we already get

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$$(13) \qquad \forall t > 0 \quad r\left(-v\left(e^{x_t}, t\right) + \left(\partial_1 v\left(e^{x_t}, t\right)\right) e^{x_t}\right) \; dt$$

$$+ \left(\partial_1 v\left(e^{x_t}, t\right)\right) \cdot e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left(x_t\right) \cdot \nu_j \left(m_j - x_t\right) \; dt$$

$$= d\pi_t$$

$$= -\, dv\left(e^{x_t}, t\right) + \left(\partial_1 v\left(e^{x_t}, t\right)\right) d\left(e^{x_t}\right).$$

Inserting (11) and (12) into (13) yields

$$(14) \qquad \forall t > 0 \quad r\left(-v\left(e^{x_t}, t\right) + \left(\partial_1 v\left(e^{x_t}, t\right)\right) e^{x_t}\right)$$

$$+ \left(\partial_1 v\left(e^{x_t}, t\right)\right) \cdot e^{x_t} \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left(x_t\right) \cdot \nu_j \left(m_j - x_t\right) \; dt$$

$$= -\, \partial_2 v\left(e^{x_t}, t\right) \; dt - \frac{\sigma^2}{2} e^{2x_t} \Delta v\left(e^{x_t}, t\right).$$

If we now choose the abbreviation $u : (x, t) \mapsto v\left(e^{x_t}, t\right)$, we obtain from identity (14) via

$$(15) \qquad \forall t > 0 \quad \partial_2 u\left(x_t, t\right) + \frac{\sigma^2}{2} \Delta u\left(x_t, t\right) - r u\left(x_t, t\right) + r\partial_1 u\left(x_t, t\right)$$

$$+ \left(\partial_1 u\left(x_t, t\right)\right) \cdot \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left(x_t\right) \cdot \nu_j \left(m_j - x_t\right)$$

$$= 0$$

and due to the recurrence of x the following partial differential equation for u:

$$(16) \qquad \frac{\sigma^2}{2} \Delta u + \partial_2 u + r\partial_1 u + \left(\partial_1 u\right) \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left((\cdot)_1\right) \cdot \nu_j \left(m_j - (\cdot)_1\right) - r u = 0$$

$$\text{on } \mathbb{R} \times (0, T)$$

for any $T > 0$. Now suppose we already know the value of the derivative D at time T as a function of x_T (this is to say that D is *European*). Then u is completely determined on $\mathbb{R} \times [0, T]$ as the unique solution to the terminal value problem

$$(17) \qquad \frac{\sigma^2}{2} \Delta u + \partial_2 u + r\partial_1 u + \left(\partial_1 u\right) \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left((\cdot)_1\right) \cdot \nu_j \left(m_j - (\cdot)_1\right) - r u = 0$$

$$\text{on } \mathbb{R} \times (0, T),$$

$$u(\cdot, T) = g \quad \text{on } \mathbb{R}.$$

One can even write down an explicit formula for u by solving this terminal value problem (17) probabilistically: For, using the abbreviation

$$L := \frac{\sigma^2}{2} \Delta + \partial_2 + \left(r + \sum_{j=1}^{N} \chi_{A_j}^{(\varepsilon)}\left((\cdot)_1\right) \cdot \nu_j \left(m_j - (\cdot)_1\right) \right) \partial_1,$$

well-known results from stochastic analysis tell us that L is the infinitesimal generator of the Markov diffusion x., and that $\partial_2 + L$ is the infinitesimal generator of the space-time process $(x_t, t)_{t \in [0, T]}$ (cf e.g. Revuz and Yor [18, Chapters VII, IX]).

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Therefore $\tilde{u} : (x, t) \mapsto \mathbb{E}^x \left[e^{-r(T-t)} g(x_{T-t}) \right]$ must satisfy the differential equation $Lu - r\partial_2 u = 0$ on $\mathbb{R} \times [0, T]$. By definition of $\tilde{u}$, we also have that $\tilde{u}(\cdot, T) = g(\cdot)$ on $\mathbb{R}$. Hence, $\tilde{u}$ solves the terminal value problem (17), which can only be true if $u = \tilde{u}$.

Thus,

$$\forall x \in \mathbb{R} \forall t \in [0, T] \quad u(x, t) = \mathbb{E}^x \left[e^{-r(T-t)} g(x_{T-t}) \right].$$

$\square$

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